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DEVELOPING UNDERSTANDING OF TRIGONOMETRY IN BOYS AND GIRLS  
USING A COMPUTER TO LINK NUMERICAL AND VISUAL  
REPRESENTATIONS

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### Declaration

I declare that the material presented in this thesis has not been presented any previous theses, except where accounts are given of my earlier research which is contained in my M.Sc. thesis, "Computer Graphics and Children's Understanding of Linear and Locally Linear Graphs", University of Warwick, 1987.



## Summary

This thesis argues that boys and girls can achieve a conceptual understanding of elementary trigonometry more successfully if they are exposed to a teaching strategy which incorporates exposure to computer graphics which link visual and numerical representations. Further, it argues that girls may gain more from this approach than boys.

The theses are tested by a classical research model involving four experimental and four control groups, with children aged 14-15. The control groups were taught by experienced teachers and the experimental groups were taught using specially designed software.

The performance of these groups is measured by a pre-test, a post-test and a delayed post-test and analysed using one-way analysis of variance. The scores of sixteen matched pairs from the whole sample are analysed using a non-parametric test. Section A of each test is designed to test for a conceptual understanding and section B involves the application of procedures which may or may not be meaningfully learnt. The scores are tested for significance using a two tailed test with  $p < 0.05$ .

After a further delay of 6-8 weeks, samples from the experimental and control groups were interviewed to view their thinking when attempting trigonometry questions.

In both post-tests the experimental subjects performed significantly better than the control subjects for section A of the test and for total scores. For section B the experimental groups performed significantly better than the control groups in all but the highest ability groups.

For post-tests, in the experimental groups the girls performed better than the boys, whereas in the control groups the boys performed better than the girls. The difference between the performance of girls in the experimental groups and those in the control groups in post-test two was significant, ( $p < 0.05$ ) for all groups, whereas for the boys, the difference was not significant for any of the pairs of groups.

# 1. Introduction, Background to the Research and Overview of the Problem.

## 1.1 Introduction

This research is primarily concerned with testing the thesis that specially developed software, linking numerical and visual representation, with an appropriate teaching strategy can facilitate conceptual learning in secondary schools. It will argue that children who experience teaching of this kind will gain a more conceptual understanding of trigonometry without any detrimental effects on their ability use procedures efficiently to answer questions associated with the topic. It will also argue that, contrary to much popular opinion, girls may be more successful than boys in acquiring concepts and applying procedures, with teaching based on using a computer to link numerical and visual representation. The background to the research will enlarge upon what led to these points of view, and the arguments are given in detail in following chapters.

The research investigates the validity of these arguments by means of a classical research method, using experimental and control groups and comparing scores after applying a pre-test, a post-test and a delayed post-test. In addition, selected members of the control and experimental groups are interviewed after the delayed post-test in order to try and gauge the effectiveness of the teaching they



experienced on their ability to relate trigonometric procedures to a visualisation of the problem being represented.

## 1.2 The reality of computer usage in secondary mathematics

As head of the mathematics faculty, incorporating computer studies, in a large Warwickshire comprehensive school from 1982 to 1986, the use of computers in secondary schools was a long standing concern. The spending of large sums of money, at least in terms of the funding available to secondary schools on computer hardware and the distribution of these resources between three sites was the focus for considerable debate during these years, and the debate continues within the school to the present. The requirements of various curriculum areas, which in the light of national curriculum requirements is more of a contentious issue than it was, and the use of hardware for administrative purpose adds further complications to the issue of appropriate resources for mathematics teaching.

In addition to the standard funding, this school had the good fortune to receive substantial private funding from a local charity. This allowed spending on computing facilities to exceed that normally associated with secondary schools. Many evenings of discussion were the precursors to crucial purchases. At the time, these discussions centred on the current state of development of computer hardware,

frequently foundering on the rock of new developments being planned by computer manufacturer at a time when decisions about current models were about to be made. Hugh Burkhardt (1985) points out that curriculum development involving advanced technology is more difficult than usual because of the "mismatch of time scales between technical change (one year) and curriculum change (ten years)" (p 148). In 1982, when the school was heavily involved in technical change this appeared to be a conservative view of the mismatch. There is no evidence that the process of technical change is slowing down, and it could be claimed that curriculum change is being speeded up by legislation and the National Curriculum Council.

The unsure position of software was a major cause for concern, with the fear that technical advances in hardware and support in maintenance was not being matched by the development of suitable school based software which teachers would find rewarding. What became evident after the purchase of a computer network was that without appropriate software being written for the specific models, the network soon became redundant as an aid to teaching anything other than computer studies. In fact after a short while the teaching of computer studies became its singular use. Other subjects within the school were barely touched by the new technology, either in the form of the network or in the use of the individual B.B.C. computers available for classrooms, complaining of lack of expertise and the infrequent



opportunities available to search for, and become familiar with, appropriate software.

Mathematicians were involved in the teaching of computer studies and more than any other department could be described as 'computer literate', yet there was little evidence to support the statement by T.J.Fletcher H.M.I. that "the implications of the current technology for mathematics is so extensive that it is difficult to present a balanced appraisal without seeming to exaggerate" (D.E.S. 1983). In fact the mathematics curriculum within the school had been unchanged and largely unaffected by computer technology, particularly in the 12-16 age range. The 'A' Level courses had the opportunity of using a few didactic programs for statistical functions and other areas of the syllabus which were available for students to use on the B.B.C. computers. However, they were not considered as being of major importance to the 'A' level teaching and many students completed the course, achieving high grades without using the computer at all. In short, the consideration given to the appropriation of resources was not reflected in curriculum or pedagogical change. The teachers in the mathematics faculty felt that they could see very little advantage over traditional methods in using computers which would outweigh the difficulties of arranging network time or the movement of stand alone computers into mathematics classrooms.

The Mathematical Association sub-committee on "Using the Computer in the Secondary Mathematics Classroom" (1987) found in their survey of 52 secondary schools that the picture described for this specific school is in fact a general one, with most computers in schools earmarked for computer studies and only 5% set aside specifically for mathematics. Further, they found a "degree of resistance to the use of computers amongst mathematics teachers: 33% of the sample never used a computer in the classroom and 39% used it very rarely" (p44)

Although there has been more interest in this specific school in the way computers can be used in the classroom, which can be traced to factors to be detailed in the next section, at the time of the research being carried out, in the Autumn term 1988, it was impossible to use the school's existing resources to teach the experimental groups, for later investment in computers had been in the direction of installation of two networks which were booked almost entirely by other subjects. Stand-alone computers had to be borrowed from other sources for the duration of the research.

In an attempt to gauge how much had changed since the Mathematical Association sub-committee report, the researcher asked 20 heads of mathematics departments from the Birmingham area, who had enrolled on courses for the management of mathematics departments, in the Spring term of 1989, how computers were used in their schools for



mathematics in the 11-16 age range. Of the twenty departments, only two had made any provision for the requirement of computer usage in the teaching in their departments and this was for one academic year only. Of the others, four believed that some teachers booked the network at some time and the other fourteen felt that there was little or no use made of computers in their departments. They recognised that the national curriculum may demand that changes be made but the overwhelming feeling of these teachers was that the school networks were heavily in demand for Business Studies, Computer Studies and Word Processing courses and that demand from other courses was increasing as provision for the national curriculum was being established. Disturbingly, many of these heads of department felt that they could see few advantages in using computers, in terms of affecting the way in which children learn, which would justify the purchase and maintenance of computers for use in mathematics.

It was clear from the outset of computer development in the school that was much to be done in terms of testing the effectiveness of using computers in facilitating conceptual understanding and communicating findings to teachers. It appears from the Mathematical Association report and this small survey that there is still a lack of appreciation of possible benefits of using computers in mathematics amongst mathematics teachers.

### 1.3. Personal research and previous experience relating to the use of computers in secondary mathematics

During the years 1984 and 1985 I had the good fortune to work with Dr. D.O. Tall in the teaching of 'A' level students using the software he had written. This became part of his research into the effectiveness of using interactive computer graphics in the teaching of calculus (Tall D.O., 1986) and an article written jointly on investigating graphs and calculus (Tall D.O., Blackett N., 1986). The use of computer graphics as part of a planned teaching strategy was seen by pupils and teachers alike to have been very successful and was reflected in the high standard of A level grades achieved by these groups. The method employed in teaching these groups was that of demonstration using the computer, dialogue between students and teacher and between students, which led to students freely using the computer to develop and test conjectures and solve problems in a meaningful way. Tall (1986a) referred to this teaching style as the "enhanced Socratic method". The software which facilitated this style of teaching and learning had the advantage of extreme flexibility, having a large capability but responding to the input of the user rather than imposing a didactic framework. As such the user required interaction with the teacher and appropriate exercises before they were able to use the computer more freely to build and test conjectures

Teaching 'A' Level frequently has the advantage of working with smaller groups of students than is usual in the 12-16 age range and students who opt for further study have already achieved some measure of success. The issue of whether computers could be used with larger classes of younger secondary age pupils to build conceptual understanding became the focus of research which was to lead to the M.Sc. thesis "Computer Graphics and Children's Understanding of Linear and Locally Linear Graphs" (Blackett N., 1987).

This research was based on the use of one computer in the mathematics classroom for part of a six period series of lessons with experimental groups, from the fourth-year pupils, on the topic of linear functions. The results were to be compared with those of control groups using matched pairs from the corresponding groups and tested for significance using the Wilcoxon matched pairs signed rank test. The software used was 'Supergraph' by David Tall (1985), and the computer was used to complement a teaching approach based on facilitating a global understanding of linear graphs which would lead to a consideration of the more sequential algebraic processes associated with this topic. After the initial experiment was completed, the research was extended by introducing the most able group from the experimental groups to non-linear functions and the concept of local linearity and on to the introduction of the



derivative. For this part of the research the pupils used 'Graphic Calculus' (Tall D.O., 1985).

The methodology involved teacher demonstration and pupil use of the computer in small groups, with exercises on the computer complemented by activities such as graph sketching. The pre-test identified areas which had been relatively unsuccessfully learned in previous teaching and the teachers of the control groups, together with the researcher identified the objectives for the teaching sessions which followed. At the end of the first stage of the research a post-test was given which incorporated questions from Daphne Kerslake's research on the understanding of graphs from Children's Understanding of Mathematics: 11-16 (Hart K.M., 1981), which helped to place the results in a national perspective.

The structure of the school, which will be described in some detail in chapter six, was helpful in designing the research, as it was suitable for establishing matched groups for a classical comparison of control and experimental groups. The same structure is used for the present research. The school had developed a policy of mixed ability teaching in the second and third years and a series of tests and end of year examinations given during these courses became the basis for the fourth year sets. As in any such process the averaging of marks over a series of tests is likely to produce scores heavily weighted by the relative success in the dominant processes being tested. In the case



of much of the earlier work in this school, and this school is not likely to be different from most other secondary schools in this respect, there is a weighting in favour of numerical calculations and early algebraic processes, with less emphasis on visualisation. We know from the Cockcroft Report, "Mathematics Counts" (1982), of the seven year difference in the range of numerical calculation skills in children at the age of eleven, and this very difference can form the basis of measures of ability for children throughout their school mathematics.

The research on linear functions furnished some evidence that there were children who had well developed spatial skills which favoured an approach based on global visualisation but who had not achieved success in the arithmetic or serialistic approach to algebra. These pupils who were in the third of four sets were moved into set one for the second stage of the research where they produced results as good as students who were considered to be of a much higher ability in mathematics, in an area of mathematics which is only usually encountered by students considered to be very able and who had opted to study mathematics as an advanced level subject.

The global approach allowed students to form conjectures about the way changing the parameters 'm' and 'c' in the general equation  $y=mx+c$  by inputting equations and observing the lines produced. They were then able to test these conjectures by sketching lines and comparing

sketches with the computer-drawn lines. The post-test revealed that the experimental groups had developed a conceptual understanding of the equations of lines and special cases, such as equations which produce parallel lines, which was superior to that of the control groups, achieving statistically significant higher scores ( $p < 0.05$ ) to the control groups which were even more markedly superior to those achieved by the wider sample used by Kerslake on the questions incorporated from her research.

The second stage of the research showed that students from set 1, together with some from set 3 had formed a conceptual understanding of linear equations using a computer graphics approach were able to use the dynamic moving tangent, which is a major feature of Graphic Calculus 1, to develop an understanding of a 'gradient function' and apply this to non-linear functions to sketch derivatives.

#### 1.4 The present research.

The present research attempts to extend the findings of earlier research by investigating the effect of linking numerical and visual representation in a topic encountered by students in 12-16 age range. The topic of trigonometry is often associated with a more algebraic approach, though it is essentially based on geometric concepts. From personal experience this topic had been thought by teachers to be a very difficult topic for

children to learn and the evidence from the cluster of tests given in the A.P.U secondary survey (1982) support this personal view, with trigonometry having the lowest mean score for any sub-category of any of the five clusters of tests. Perhaps this should be unsurprising for a conceptual understanding of trigonometry is dependent on an understanding of ratio, which when applied to similar figures was shown in "Children's Understanding of Mathematics" (Hart K.M., 1981) to be too difficult for the majority of children questioned, with the test items being "some of the hardest on the test paper" (p100). The report went on to say that,

"Using ratio to share amounts between people 'so that it is fair' seemed to be much easier than dealing with a comparison of two figures. In enlarging figures there is the danger of being so engrossed in the method to be used that the child ignores the fact that the resulting enlargement should be the same shape as the original", (p101).

This point is essential to the computer based approach in this research as it was in previous research; the computer attempts to overcome the problem of children focussing on local problems involved in the construction of particular geometric processes by allowing the student to perceive visual patterns which can be constructed from a large number of examples without the student having to become engrossed in the method of construction.

There is too, with this topic, the introduction of new words 'sine' and 'cosine' which must be linked to



mathematical concepts without reinforcement from any other usage outside of the classroom. The word 'sine' having moved through the ages from the Greek word for chord through Arabic, where only the consonants were written and mistranslated into Latin in its present form as a word meaning 'cape' or 'bay' or 'bosom'!

The expectation of pupil performance in trigonometry at G.C.S.E. level is typically that:

1. at the foundation level, target grade F, grades available E, F, and G, no trigonometry is required at all

2. at the intermediate level, target grade D, grades available C, D, E and F, students should be able to apply the sine, cosine and tangent of an angle to calculate a side or angle in a right angled triangle and to problems in two dimensions.

3. at the highest level, target grade B, grades available A, B, C and D, students should, in addition to the demands of the intermediate level, be able to apply the sine, cosine and tangent of an angle to problems in three dimensions.  
(Extracted from Southern Examining Group Syllabus, 1992 examinations)

The national curriculum document "Mathematics in the National Curriculum" (D.E.S., 1989) places trigonometry in 'Attainment target 10: Shape and Space' with "using sine cosine and tangent in right-angled triangles in two dimensions" at level eight, "calculate distances and angles in solids using plane sections and trigonometric ratios",

and "find sine, cosine and tangent of angles of any size", at level nine, and "sketch the graphs of sine, cosine and tangent functions for all angles, generate trigonometric functions using a calculator or computer to interpret them, and use sine and cosine rules to solve problems including simple cases in 3D" at level ten, the highest level.

The national curriculum demands more than present G.C.S.E syllabuses at the highest level, but by placing the introduction at level eight, assumes like the G.C.S.E. foundation level, that some children should not meet the topic at all, and only introduces the computer as a means of interpreting functions at the highest level.

Trigonometric functions play a large part in advanced level mathematics courses as they do in Physics courses, where wave motions are investigated. In addition, Newtonian mechanics relies heavily on vectors, often demanding a speedy, efficient recognition of trigonometric relationships and corresponding calculations. Children who have no understanding of trigonometry are denied access to much of the advanced work in mathematics and physics, so increasing the number of pupils who develop a conceptual understanding of trigonometry would increase the possibility that more students could be encouraged to study these subjects at a higher level. The research investigates how successful a computer graphics approach can be in building such a conceptual understanding in children from who would

sit the highest level of G.C.S.E papers to those who would sit the foundation level papers.

The research on linear and non-linear graphs could be said to justify a computer graphical, global visualisation approach to a topic which has as its end product an understanding of 'pictures' or graphs. It was a matter of some concern that pupils should be able to use a visualisation approach to mathematics but that they should also be able to be flexible enough in their thinking to move between algebraic or numerical approaches and overall visualisation when appropriate. The topic of elementary trigonometry allowed the research to progress to the development of software which had numerical and visual representation designed to encourage students to perceive patterns and formulate conjectures in one mode of representation and compare these with the results displayed in the other mode.

The aim of the lessons would be to allow students to build a conceptual frame for trigonometric processes, but also to allow students to develop a facility for visualisation which would allow them to visualise problems, possibly by reinterpreting diagrams to produce more realistic representations of problems, and to use these as a basis for the numerical or algebraic processes.

Students who had learned in this way would, it was hoped, be able to recognise the definitions of sine, cosine and tangent of an angle from visualisation processes but



would also be able to use them more effectively than students who had attempted to learn algebraic procedures without the benefit of a computer.

The present research was planned to test the thesis that students exposed to a computer approach designed to link numerical and visual representations in trigonometry would achieve a greater conceptual understanding but would also be able to perform the procedures associated with the topic as well as students who had been taught using a more traditional approach. It would also incorporate a delayed post-test to identify what changes in performance for both control and experimental groups were associated with a longer time gap between teaching and testing. The null hypothesis will be that there is no difference in performance between the control and experimental groups in parts of the test concerned with conceptual understanding or in the parts associated with well practised procedures.

The introduction of taped interviews after a further delay would allow the researcher to gain some overall view of the ways in which children from both experimental and control groups approached typical problems, in particular whether they were able to visualise a problem as well as accurately perform the associated procedure. The research design and structure is described in detail in chapter six.

No attempt had been made in previous research to assess whether boys and girls differed in terms of the

possible benefits to be gained by incorporating computers into their teaching, the researcher had gained an impression that girls were particularly capable of linking the visualisation process to the calculations involved in plotting graphs. In research on using a computer to create a global view in the teaching of algebra, Michael Thomas, (Thomas M., Tall D.O., 1988) was to observe that girls performed better than boys in the post-test, having performed less well on the pre-test, though an examination of a possible relationship between performance and gender was not initially intended.

Statistics on relative performance of boys and girls in mathematics are available; some are given in detail in chapter five, and there are various theories on differences in cognitive attributes associated with the two sexes which are also discussed in some detail in the same chapter. The result of this analysis is an argument that girls may have more of an advantage than boys from teaching where a computer is used to link numerical and visual representation.

By marking the pre-test and both post-tests to give the girls' and boys' scores separately, it is possible to compare experimental and control groups for boys and girls and to compare boys and girls in the same group in order to assess whether the control or experimental treatment would be associated with success for one sex more or less than the other. The null hypotheses will be that there will



be no difference in performance between girls in the experimental group and girls in the control group, and there will be no difference in performance between boys in the experimental group and boys in the control group.

The possible effects of using computers to link visual and numerical representations will be discussed in later chapters by linking the information processing approach in cognitive psychology to mathematical education, cognitive development, and research into the way the brain functions.

## 2. An Information Processing Approach to Learning

### 2.1. Preamble

This section explores an information processing approach to the ways in which learning takes place, and this will be compared in the following chapter to the theories of psychologists who have been influential in the field of education. Such terms as "concept", and "understanding" will be examined in the light of these theories and in the later chapter these will be linked to discussion on the human brain and aspects of meta-cognition, so that the central thesis behind this research can be placed in a theoretical perspective.

The consideration being given to theories of mathematical development follows the considerable influence psychologists such as Piaget and Skemp have had on the way in which we view the mathematical development of children and, in consequence, on the ways in which mathematics can be presented to facilitate learning. However, in order to present the theory upon which this research is based it is necessary to examine the way in which cognitive development can be expressed and analysed in information processing terms. Although there may be significant differences between the development psychology of those following a Piagetian position and that described by information processing, it is

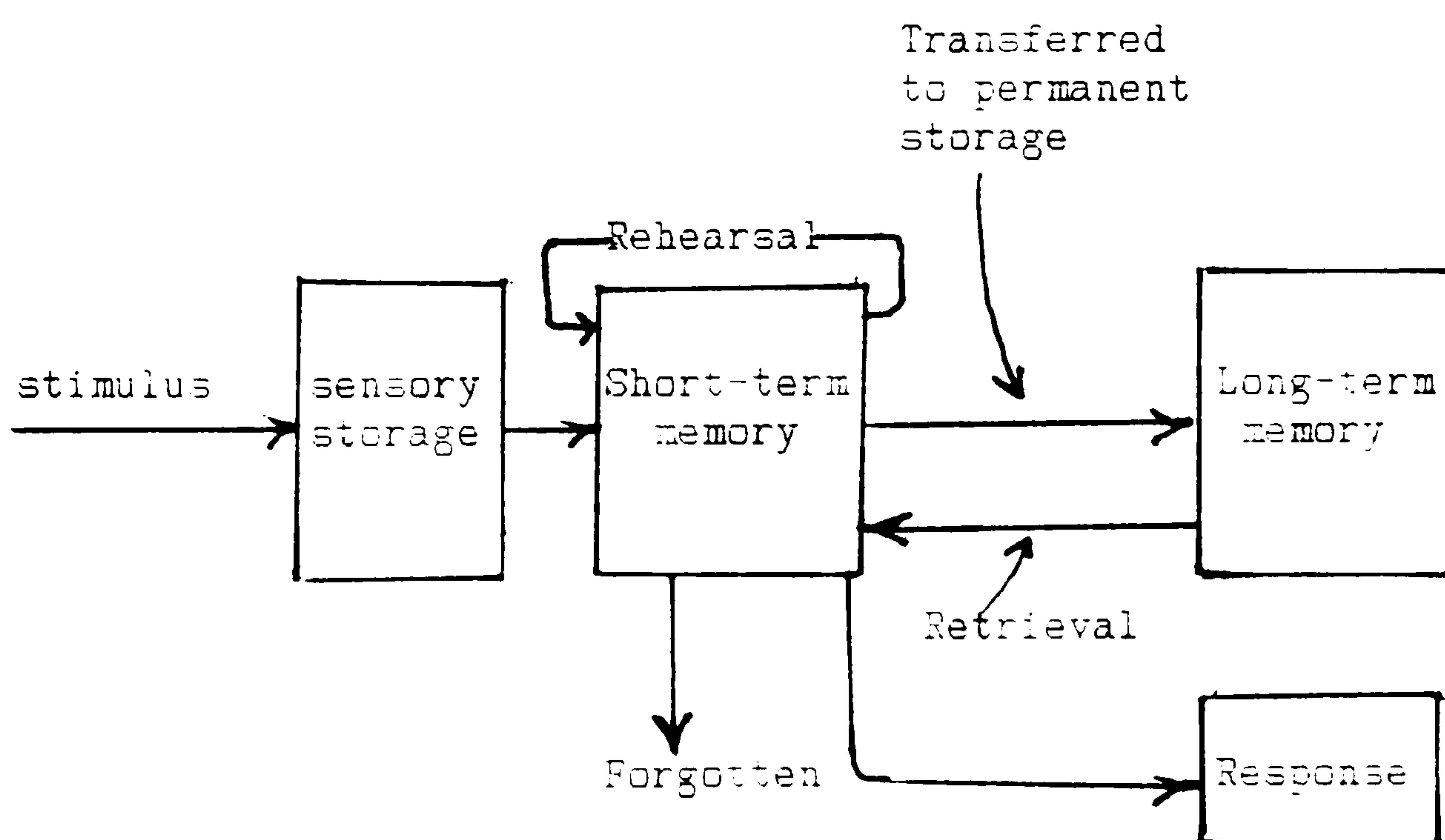
possible to relate much of the content of these theories to information processing models.

## 2.2 Introduction

Cognitive psychology is the study of how knowledge is stored, transformed and used. It can be approached from several varying points of view which may depend on philosophical positions on the nature of reality as well as findings based on neurological research or psychological theory and experimentation. From the information processing standpoint the study of knowledge assumes that information is variously processed through a series of mental activities in which functions are performed before the information is available for further processing in subsequent activities. These stages would include the detection and interpreting of information, memorising and utilising information from memory, and higher order cognition involving reflection and problem solving. Central to this research is the proposition that in concept development in mathematics too little attention has been paid to the forms in which information is presented and the effects of modes of presentation on the perception of patterns which underlie the concept development.

Descriptions of the processing of information frequently result in cognitive models which attempt to show the relationships between stages in the processing and present

new hypotheses about the processes involved. These models can only be seen as metaphors and the limitations on these are obvious when the complexity of the activities being modelled are examined. Nevertheless, models of processes can be a useful way of communicating theories on the various stages of processing and such models were valuable in considering the possible effectiveness of the teaching methods used in this research. A simple cognitive model suggested by Robert L. Solso (1979) which itself is a modification of the earlier Waugh and Norman (1965) model illustrates the flow of information through the principal processing components. Though it is unlikely to satisfy all the demands made by cognitive psychologists it serves as an adequate basis for discussing the information processing approach.



(From Solso R.L. 1979. P18)



Elaboration on each of these stages and processes, with respect to the teaching and learning of mathematics will aim to give an overview of the theory upon which this research is based. However, the complexity of theories of memory and the other aspects of information processing is such that any attempt to give an overview will inevitably greatly simplify or omit some aspects of the accumulated knowledge.

### 2.3 Detection of sensory signals and perception.

Phenomena from the external world are detected by means of the sensory system with each of the sense organs converting incoming stimuli into neural energy. For a very brief period of time the sensory systems are able to store the information before further processing is employed. This briefly held store can be visual or auditory, the visual memory now being generally referred to by Neisser's term (1967) "iconic memory" and the auditory form is generally termed the "echoic memory". Even at this initial stage there appears to be a difference between the way in which these two forms of information are processed with the echoic memory being slightly more persistent than the iconic memory and facilitating the processing of sequential information. The ability to recall the final digit in a number received a digit at a time auditorily more successfully than when the digits are read appears to be a consequence of

the echoic memory retaining the information long enough to be recalled.

These sensory stores appear to provide the facility for selecting appropriate data for further processing whilst briefly holding the original data. Thus the limited ability of the higher order cognitive systems to process information is not overwhelmed by the astronomical amount of information being sensed at any one time.

The process of perception is that of relating the detection of stimuli to previously held knowledge of the world held in long term memory. At the sensory level this information is very specific, whereas at the level of interpretation the information appears to be coded as abstract representations of reality. Changing views of the world are thus determined by the integration of what we sense specifically and what we know as abstractly coded previous experience. Some data or information will depend simply on the nature or strength of the stimuli, with little need for interaction with long term memory, but it is impossible to consider perception without considering the conceptually driven processes, where the perceiver has already formed some kind of expectation of the information which is to be encountered. Of particular importance to understanding how mathematics may be learned is pattern recognition, which appears to be determined jointly by the information available to the senses and the knowledge already coded in long term memory.

In this research the ability to recognise patterns was considered to be vital to the learning process and this recognition process requires some exploration.

Paivio (1971) makes the important point that words which are heard or read are serially processed, whereas with pictures the processing seems to take place all at once. His simple experiment involving presenting pictures and words to subjects very rapidly revealed that more pictures than words were recalled but that the subjects were able to reconstruct the order in which the stimuli were given much better for words than pictures. The ability to classify visual stimuli into patterns which can be stored, processed (that is moved into short term memory for the purpose of further cognition) and when in long term memory able to be recalled for further pattern recognition, is a vital part of the learning process. Several prominent theories exist which attempt to explain the phenomenon of pattern recognition and these can be summarised as follows:

1. The Gestalt view that pattern recognition is based on the whole pattern of the stimuli, with parts of the configuration having meaning from their membership to the whole pattern.

2. Bottom Up Processing, where the parts of the pattern initiate the recognition procedure which when summed leads to the recognition of the whole.

3. Top-Down Processing, where recognition of whole parts leads to recognition of the component parts.



4. Template matching, where a match is made between sensory stimuli and some corresponding internalised mental representation.

5. Prototype Recognition, where the pattern recognition occurs when the the sensory stimuli are matched to some internalised abstract or idealised mental pattern.

6. Feature Analysis, where the pattern recognition occurs when prominent, simple, features of the incoming stimuli are analysed.

Gestalt theory, is considerably older than the other theories. The work of Gestalt theorists such as Wertheimer (1945) which stressed that pattern recognition was largely a natural function of then stimulus itself, depending on such factors as proximity of elements, similarity of elements and continuous smooth directions may represent the parallel processing or "all at once" recognition of pictures to which Paivio refers. However there appears to be some difficulty in relating this theory to the recognition of complex patterns which depend on the identification of elements within the pattern. The theory of top down or bottom up processing has become more associated with the relationship between the whole pattern and its constituent elements. Nevertheless, there are similarities between top down processing and Gestalt psychology in the emphasis on perception of the whole pattern leading to an analysis of the constituent parts.



Bottom up processing would suggest that the perceiver links the constituent parts of the stimulus until some recognisable pattern emerges; top down processing would suggest that rather than examining the constituent parts to recognise a pattern, the pattern is identified as a whole and the processing takes place in analysing the constituent parts. Palmer (1975) has suggested that under most circumstances the bottom up/ top down procedures take place simultaneously with the scanning of parts being more efficient when the whole pattern is viewed in a recognisable context. His research into the way subjects were able to recall objects after being shown a picture of a contextual scene showed that the recall of objects linked to the contextual scene was significantly better than recall of objects, equally commonly found, but unrelated to the contextual scene. The question remains of how the pattern, whether approached by top down or bottom up procedures is recognised.

The simplest model of pattern recognition is that of template matching, where as a result of previous experiences the concepts stores in long term memory can be matched to incoming stimuli, in a similar way to the ways in which bar-codes are read by light pens and matched with computer stored information to allow the codes to be understood; every bar has to have appropriate computer stored information associated with it or the computer can not recognise the pattern. Whilst it appears apparent that there

must be some form of matching between things perceived in 'external reality' and, on some level of abstraction, an internal representation of previous experiences in long term memory, a literal interpretation of template matching presents some difficulties. The most obvious of these is the need for countless millions of templates to correspond with the varied forms which subjects can recognise.

In terms of teaching mathematics this is a particularly important point. Students will be expected to recognise shapes, families of equations and patterns in graphs even though in each case they may differ from the particular examples previously encountered.

An alternative model which incorporates the theory that external reality is matched against some form of internal representation, is that of 'prototype' formation. This theory would suggest that the long term memory can store a prototype of a pattern where the important factors which make up the pattern are abstracted. Stimuli are matched to the prototype by analysis of the features. An experiment by Posner, Goldsmith and Welton (1967) illustrated the way in which subjects were able to classify distortions of prototypes given in the form of patterns of dots, without seeing the original prototypes. The subjects were able to classify distorted patterns of dots into a common category, whilst other distortions based on other prototypes were sorted into other categories. The three prototypes used were familiar shapes (a triangle, and the letters F and M). When

presented with new patterns to sort, based on old distortions, the prototype patterns which had never been seen before, and new distortions, the subjects were able to classify the old distortions and the original prototypes, with 87% accuracy. The new distortions were classified less well. It appears that the subjects had learned something about the prototypes without seeing anything other than distortions.

In a similar experiment by Peterson, Meagher, Chait and Gillie (1973), the 'meaningfulness' of the prototype shape was taken into account. (In this experiment 'meaningfulness' is clearly seen as familiarity or previous experience of the prototype). Their results indicate that prototypes and minimally distorted test patterns of highly meaningful configurations were more easily identified than meaningless prototypes and minimally distorted patterns. However, where the degree of distortion was great the opposite was true: the highly meaningful prototype was less often identified with the distortions than was the low meaningful prototype. This has clear implications for teaching mathematics: the amount of variation of particular very 'well learned' prototype which will be recognisable as being a similar pattern, or distortion of the original, will be less than for prototypes which are still developing their meaningfulness. In particular the child who associates a 'well learned' diagram with a mathematical routine may be



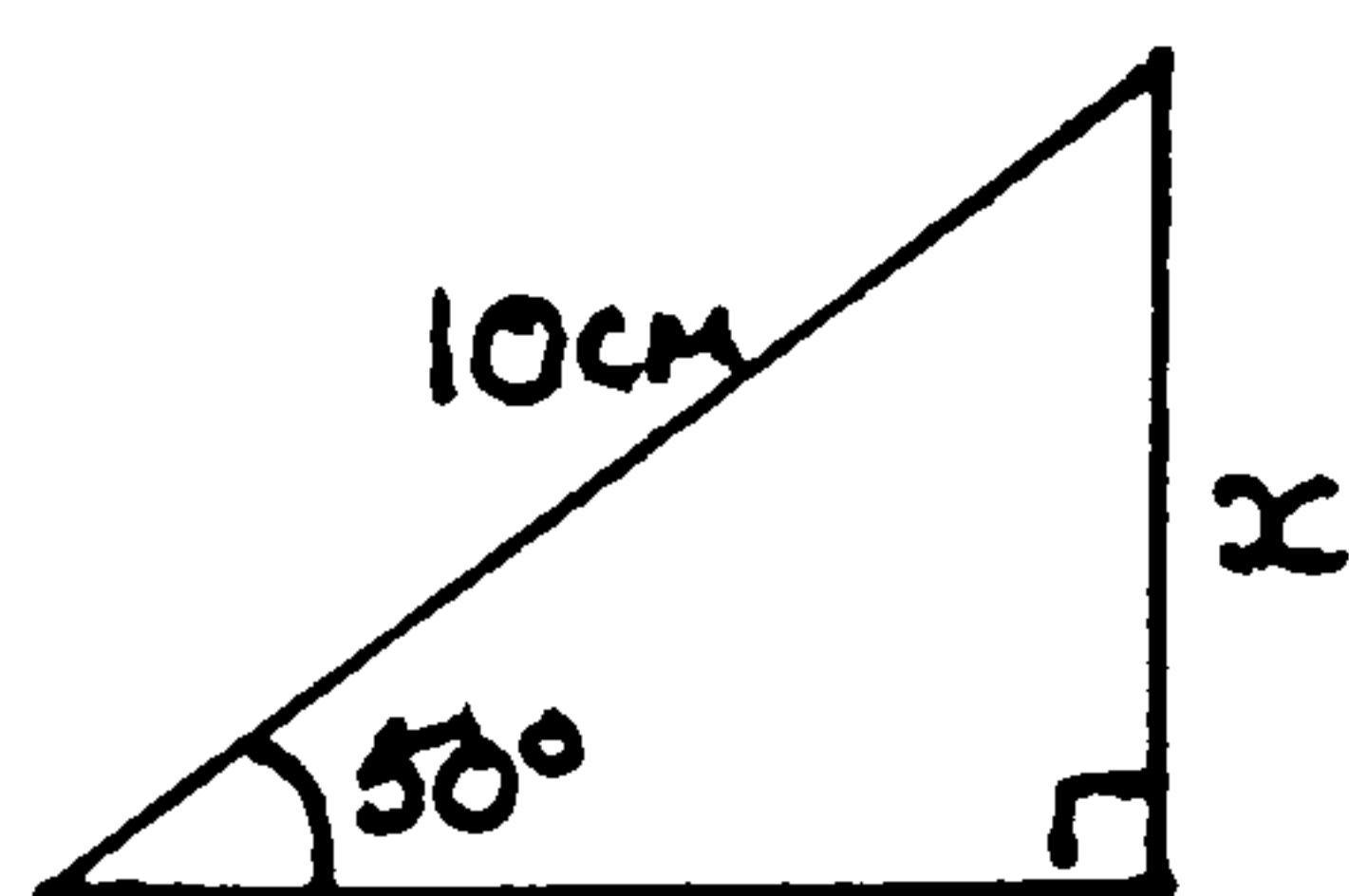
resistant to variations on the original diagram cueing a similar response.

When trying to recognise patterns from visual stimuli the notion of feature analysis gives an interpretation of initial processing which supplements rather than contradicts the theories described above. Feature analysis suggests that the kind of processing associated with bottom up/top down processing or prototype matching is preceded by an initial step where incoming stimuli are identified according to their simpler features. This has been supported by experiments which note the eye fixation of subjects when exploring stimuli. (Mackworth N.H., Morandi A.J., 1967). The eye movements indicate a scanning technique, with more time being spent on features carrying most information.

It would appear that primary recognition of features would then depend on possible recognition, by either template or prototype, before the whole pattern is processed. Thus the pattern recognition is seen as a series of 'matching' processes.

The presentation of information in mathematics education can itself be seen in terms of a form of pattern recognition exercise, with feature analysis, prototype or template matching and bottom up or top down processing taking place. The role of the diagram in mathematics is very often the way in which we can focus on salient features and recognise familiar patterns. In elementary trigonometry the features

we wish to recognise are very often the lengths of particular sides, presented as a numerical value written close to the side, in association with the value of an angle, again given as a number written in the angle. This information is then ready for further analysis. The diagram, a triangle with sides and an angle marked is matched with a prototype and the essential features are noted, but the geometry of the triangle itself is often redundant once the features are recognised. Processing is then often carried out only with the numerical or algebraic information which were features of the diagram and the information processed is in numerical or symbolic form. In the diagram below, the essential features to be noted are the angle marked as  $50^\circ$  the length of the hypotenuse and the side marked  $x$ .



For many children this information is then processed without further reference to the diagram or to any more realistic visualisation of the triangle described by the information. This research will propose that in order for the child to gain a conceptual understanding of trigonometry the recognition of geometric patterns and numerical patterns formed by a dynamic process are more valuable than merely extracting numerical or symbolic features. Further



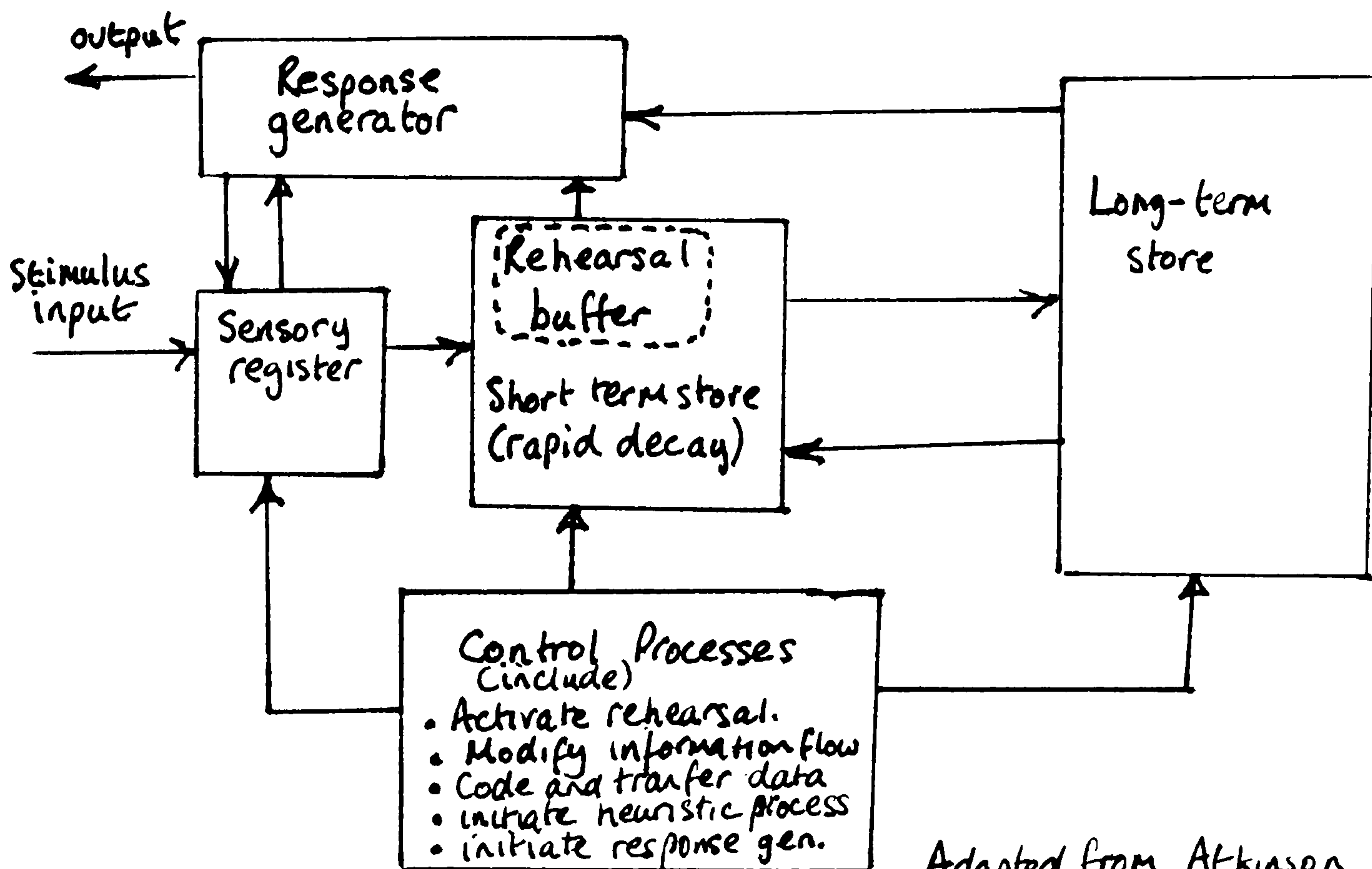
discussion on this issue will occur in the following chapter when Fischbein's view of intuition is examined, and this is key point to be developed in the interviewing stage of the research. However, the ways in which the information is further processed is vital to the consideration of how mathematical activities can be carried out.

#### **2.4 Short term memory.**

The dualist concept of memory suggests that between the sensory register and the vast memory store which is long term memory there is a short term store where information is registered and either lost or processed before storing in long term memory. Atkinson and Shiffrin (1968) propose that this short term store is a 'working' area where information decays and disappears rapidly and that the information may not be in the same form as that in the original sensory register. This short term store is of very limited capacity so that information must of necessity be transformed into long term memory or decay. They suggest that rehearsal could maintain information in the short term store for longer periods and increase the probability that it is transferred to long term memory. The Atkinson and Shiffrin model views the long term memory as relatively permanent, though susceptible to some decay, but information can be inaccessible because of interference of incoming information. An important aspect of this model is the

importance given to the control of the subject of the movement of information from one store to the other, in particular the conscious or unconscious control of the flow of information in and out of the short term memory store. This would suggest that there must be some form of executive control system which the subject uses to organise this flow.

The Atkinson Shiffrin Model



Adapted from Atkinson and Shiffrin (1968)

Baddeley (1982) supports this notion by including in his subsystems of the short term memory as a 'working' memory, the notion of a central executive which the subject uses to control the various functions of the memory. He further suggests that this central executive may have more than one component.

I would suggest that the component we have so far referred to as the central executive almost certainly has at least two sub-systems itself, one of which is devoted to memory, the other

being largely concerned with conscious attention; it is this latter system that is probably responsible for controlling both the central processes of memory and the other slave systems. (p 189)

Similarly, Gagne (1977) postulated an executive process or processes which were superordinate to the common processes of attending, remembering organising and storing information and which signal when the events should operate. This being the case, it is likely that students will be able exercise some control over the selection and processing of information in short term memory. Information within short term memory appears to be primarily coded in auditory form, that is even when the information is sensed visually, a word or number, it is held in short term memory by auditory rehearsal. However, there is convincing evidence that short term memory can be coded visually. Roger Shepard (Shepard R.N., Metzler J., 1971) of Stanford University has made a study of "mental rotation" of visual stimuli in memory. In his experiments subjects were given two pictures of clusters of cubes, the first being a rotation of the second, with the degree of rotation varying from 0 to 180 degrees. The subjects were asked to decide if the second pattern was the same as the first apart from being rotated and the responses were timed. It was found that the time taken to respond was a linear function of the degree of rotation, a small rotation being quickly judged, a large rotation taking



correspondingly longer. Shepard concluded that the subjects internal visual representation was such that the images required one second for every fifty degrees of rotation. It is clear then, that in mathematics we may be neglecting powerful working memory attributes if we only stimulate visual to auditory form for numbers in working memory, and fail to utilise the potential for visual imagery. In the example of the triangle containing information for processing in trigonometry, it is possible that the visual imagery of the triangle itself is capable of being transformed; for instance, the corresponding change in the length of the sides when the given angle is changed can actually be visualised rather than simply calculated. Given Baddeley's view of the central executive and the importance given in the Atkinson Shiffrin model to the power of the subject in selecting information to be used in the working memory, it appears that a subject could, indeed, be able to select either visual imagery or numerical forms and move between the two codes in working memory.

A central plank in the Atkinson Shiffrin model is the importance of rehearsal in processing information from short to long term memory. However, Craik and Lockhart (1972) propose that an alternative view with a model of levels of processing. In this model incoming stimuli are analysed at various levels, starting with narrow sensory analysis and proceeding to deeper semantic, abstract analysis. The stimulus may be processed at shallow or deep stages



depending on the stimulus itself and the time available. In the initial analysis stimuli may be subjected to feature analysis then at a deeper level by means of bottom up or top down processing for pattern recognition, then at a deeper level again by stronger association of meaning with information held in long term memory. Memory, in this model is very strongly linked to the attention given to the stimulus and the depth of processing. Experiments by Craik and Watkins (1973) designed to test whether well rehearsed items of information are recalled more than information processed at a deep level show that maintenance rehearsal, that is repetition in order to keep the information at a 'working' level, does not necessarily improve memory in the longer term, whereas information which can be elaborated into meaningful association leads to better recall. The theories of Ausubel and Skemp, which will be considered in the following chapter have strong links to this aspect of information processing and is particularly relevant to the view that mathematical procedures are more likely to be transferred to long term memory if they can be linked meaningfully to patterns already established in a meaningful way. In this research the underlying patterns are to be perceived as geometric and numerical links and these are the foundations which make the procedures meaningful.

Baddeley (1982) argues that the evidence from brain damaged patients strongly supports the notion of different memory systems. Patients suffering damage to certain areas

of the brain around the left cerebral hemisphere have very poor recall of very recent events whilst able to recall information processed much earlier. Other patients with damage to temporal lobes or the deeper structures of the brain, such as the hippocampus can recall recent information normally but suffer from long term amnesia. Though the Craik and Lockhart theory disposes with the idea that maintenance rehearsal is the only method of transference between short term memory and long term memory, it does not break down the notion of the short term memory, for it supposes the existence of a short term memory system which performs the processing at the various levels. Robert Solso (1979) suggests that the Atkinson Shiffrin "boxes" model, "would also seem capable of accommodating different depths of processing by the addition of several minor subroutines" (p170).

The short term memory is limited in the amount of information it can store to around seven units of information (Miller G.A., 1956). These seven units can vary from seven digits or letters to seven words or seven phrases. The process of 'chunking' describes this phenomenon. The memory store is modelled as a number of slots; letters can occupy a slot each but if chunked together to form words then each slot is occupied by a word, if words are chunked as a phrase then each slot is occupied by a phrase. In this way the very limited short term working memory can process large amounts of information quickly.



## 2.4 Long term memory and higher order cognition.

A study of memory structure, forgetting, and retrieval of information from long term memory, represents far too large a subject to be covered adequately in this synopsis of the parts of information processing theory particularly relevant to this research. However, aspects which relate strongly to the previous emphasis on pattern recognition in perception and the link between visual and semantic codes in working memory should be examined. In particular, higher order cognition, with which a great deal of mathematics is concerned, will be viewed in these terms.

Long term memory can be seen as the repository of all memories which are not currently being used but are retrievable. This information can be episodic, that is memories about events and when they occurred, or knowledge and skills based on semantic links between information.

Information in long term memory is coded (at least) acoustically, visually and semantically; we are able to recognise sounds, identify objects by sight or by description. This store of information is undoubtedly organised in some way, and this organisation is often modelled on a network theory, with specific items being recalled by entering the network and following lines based on visual recognition, semantic links or temporal links. Bower (1975) suggests that new information entering long term memory results in a reconfiguration of a part of the



network, but that this reconfiguration operates within the pre-existing structure, and this pre-existing structure can set off a chain of inferences about the new situation. The specific networking models being proposed by various experimental psychologists are not necessarily relevant to this research, but the idea of stimuli acting as cues for recall within an organised network of semantically, visually or temporally linked data is important.

The term 'concept' is much used in mathematical education, as it is in information processing theory. In both, a 'concept' may be defined in terms of common critical features which define a class of objects, and rules which relate these features. Thus at a simple level the concept of a circle may be established if the common features of shapes labelled circles are deduced and examples can be distinguished from non-examples. In higher order concepts, such as quadratic equations, the examples and non-examples may be can only be identified if the relationship which governs membership of the example set is more closely examined. This higher order concept is obviously dependent on the lower order concept of 'equation'.

The influential book "A Study of Thinking" (Bruner, Goodnow and Austin, 1956) suggests that the first stage in concept attainment is the selection of a hypothesis and then the testing of data for examples or non-examples of the concept being hypothesised. The particular strategies employed include "successive scanning" and "conservative

focusing". In the first instance the subject begins with a hypothesis and maintains it if successful, if it is unsuccessful the hypothesis is changed. In the example above, the subject may hypothesise that quadratic equations are those which contain three terms, which would be rejected when considering or reconsidering  $x^2 = 1$ . In conservative focusing the subject formulates a hypothesis based on a positive instance then changes one feature at a time noting which change denotes an example and which a non example.

In information processing models the inter-relation between the short term working memory, with chunks of information, and long term memory is such that both techniques may be used depending on the strength of the features in the stimuli. The conservative focusing may result in a decision tree, which results in the hypothesis being tested by scanning a range of stimuli. The Atkinson Shiffrin model emphasises the control part of working memory as being important in selecting strategies, but essential to both strategies is the existence of examples and non-examples of the concept to be tested. It will be argued that a major benefit of the computer graphics approach is the speedy generation of examples, which, as the user does not have to concentrate on the construction can assist the strategies for concept acquisition.

In mathematics we refer to 'concepts' and 'conceptual learning' in a way which is more expansive than the definition above. Conceptual learning would involve the



ability to apply and relate concepts in such a way as to produce meaningful results. Gagne (1966) attempted to distinguish between the more simple 'concept' as it has been defined above, and the more general use of the word when used in the sense of, for instance, the 'concept of trigonometry'. He introduced the term 'principle' to include the demonstration of relationships between two or more concepts, (defined as above). This relationship would frequently involve 'situational mediators', such as angle, or length, and 'transformational mediators' such as multiply or divide. However, it would be true to say that Gagne's use of 'principle' has not become part of popular terminology and 'conceptual learning' can be taken to include the identification of concepts and the demonstration of Gagne's 'principles.

Robert B. Davis (1984) introduces new terms from cognitive science, which expand upon Bower's 'chain of inferences' set off by new information entering long term memory, which are particularly pertinent to a description of mathematics teaching and learning. Davis refers to linked concepts in long term memory as a "knowledge representation structure" and within this structure he defines a 'procedure', 'sub-procedure' and 'super-procedure'. A procedure is a sequence of instructions recorded in long term memory which was recorded to accomplish some specific result. It has, according to Davis, a name or label to allow it to be identified and slots which can accept new data into



the list of instructions. When a procedure is begun, it copies the instructions into working memory at an appropriate rate and the activities are actively carried out. Sometimes the procedure will arrive at a step where another procedure is needed in order to complete the instructions and this second procedure is identified and copied into working memory, the steps are activated and then the original procedure is brought back into play. The first procedure is then a "super-procedure" and the second is a "sub-procedure". These linking procedures will, in everyday events, be of great complexity; in mathematics they may involve very complex interactions between super and sub procedures. However, procedures may be 'rote learnt', moving into working memory as an automatic response to incoming stimuli, and may not necessarily link to other procedures or to other parts of the knowledge representation structure, or they may be meaningfully learnt, where the same routinised response is linked to other parts of the knowledge representation structure

Davis describes "visually moderated sequences" as a series of written visual responses which break up an overall super-procedure. Each written form provides a visual cue to a new sub-procedure. This notion of the visual cue to a semantic procedure represents a different view of the serial thinking which may be imagined to be happening when students are following algebraic processes. It means that the very design of the written response is more important than mere

presentation skills, for the 'picture' rather than the semantic content can be the cue to the next stage of the serial process. It will be argued that children who fail to establish conceptual learning can rely on such procedures, where the actual form of the visual cue provides the routinised response, rather than a process of algebraic thinking.

There is a great deal of evidence from the experiments of Paivio (1971) that information is represented in long term memory by two distinct codes: the imaginal and the verbal. Though these two codes may overlap in the processing of information, there is likely to be a greater emphasis on one rather than the other. A familiar picture may be coded both imaginally and verbally but the verbal code may be aroused later than the imaginal because an extra transformation is required. Abstract words, Paivio suggests, are coded verbally only, whereas concrete words are coded in both codes. Clinical observations of neurologically damaged patients tends to support this view, with left hemisphere brain damage being associated with verbal memory problems and right hemisphere damage being associated with memory of visual material. Further reference to right and left hemisphere activity will be made later in this thesis.

That imagery and verbal information are processed differently may be seen as an inevitable result of the kind of stimulus, with the subject having little control over the processing. However, Bower (1970) has shown that when



subjects are given paired associate experiments. that is tests of recognition and recall of pairs of words, for which the picture and the word are given to the subject, the rate of recall can be greatly increased when the subjects are encouraged to visualise relationships between the two words in terms of their pictures. (A subject asked to recall 'piano' with 'cigar' would have been asked to mentally construct an image of a piano smoking a cigar for instance). Bower is thus suggesting that the subject can "relationally organise" information in imaginal and verbal codes.

Shepard (1968), when describing the experiments referred to earlier, in discussion of short term memory, has concluded that some information can be held entirely in terms of visual imagery, with no corresponding verbal coding. Subjects could recognise visual stimuli which had no appropriate verbal coding, place this information in imaginal terms in short term memory and carry out mental processes. Shepard has not advocated that the imaginal processes are exactly isomorphic to the actual processes of rotation or reflection, but introduces the idea of "second order isomorphisms", that is the objects are not directly or structurally represented in our brains, but the way the internal representations work is very similar to the way the external representations work. Shepard's work is making the point that the visualisation processes undergone by children in dealing with some aspects of mathematics will be closely linked to the perceptual patterns which gave rise to the



concepts being established. Asking children to 'imagine' certain geometric transformations may be asking the impossible, if they have not previously perceived such transformations.

Davis's description of visually integrated sequences represents a link between visual imagery, perception and verbally coded information in mathematics, but he also suggests that we can internalise the visual cues to provide an integrated sequence. Thus, at a later stage of learning, the learner has represented the visual cues in long term memory and the complete super-procedure can be completed without recourse to writing. From the research described above it would seem likely that the sub-procedures can themselves consist entirely of mental imagery.

The knowledge representation structure relating to stimuli is likely to consist of much more than procedures to accomplish specific results, but a vast range of related concepts associated with the original stimulus. The terms 'schema', 'scripts' and, more recently in information processing theory, 'frames', are used to refer to these inter-related concepts. The frames associated with particular cues in the stimuli gives background information and contains slots or variables which need to be filled from the incoming information in order to make the new information meaningful. If the information is not present then the frame may contain default variables (assumptions), and the frame is ready for further processing (or thinking).

If a key variable can not be filled from either the information representing the input data or from a default variable then the frame can not be operated on further. This notion has strong links with the principle of 'accommodation', which will be examined in the following chapter, for there is also the implication that if incoming data cues responses which appear to be contradictory, then no further processing can take place until the contradiction is resolved. For example, incoming data in a trigonometry question to the effect that a particular 'bearing' for a ship is 128 degrees for a particular distance and asks for the distance travelled south and east, may cue a trigonometric frame, which can not accept an angle of 128 degrees.

The efficiency in selecting an appropriate frame appears to depend on a good deal on the context in which the incoming data is received. Baddeley (1982) quotes examples of students' comprehension of passages increasing greatly when the subject of the passage is given beforehand, but not when it is given after reading but before being tested for comprehension. (p82) This appears to support the Craik and Lockhart argument, to which reference was made earlier, in the sense that the context of the passage allowed a deeper level of processing by making it more meaningful. Students of mathematics presumably bring expectations into the mathematics classroom which may enable them to select frames appropriate to the information received. It is likely that



previous experience of a requirement to establish procedures which may not necessarily be meaningful, particularly if they have been associated with success, will lead to the expectation that new teaching will provide the context for more procedures. Baddeley's examples support the idea of 'advance organisers', which will be discussed in the following chapter.

The kind of response engendered by particular inputs of data will depend on the kind of pattern recognition being employed as the short term or working memory processes the information. At a simple level a picture may be recognisable almost instantaneously, other more complicated stimuli may require feature analysis and some top down processing before an appropriate procedure or frame is recognised. Others may require a bottom up process before the stimuli itself is made meaningful. We may consider, for instance, how a sixth former may approach the question of how to solve an equation of the form  $8\sin^2 x + 2\sin x = 1$ .

The initial feature analysis may establish that the equation is in terms of trigonometrical functions, but also that it has some similarity to the quadratic equation  $8x^2 + 2x = 1$ , which from several years of previous experience is likely to exist as a prototype recognisable as similar to  $8x^2 + 2x - 1 = 0$ . Two frames are brought to bear from the initial feature analysis of the input. If the quadratic equation frame is unavailable, the trigonometric frame will be inadequate for the subject to proceed further, but if the



quadratic frame is not linked to a suitable frame for trigonometry then the procedure for solving quadratic equations, either as an integrated sequence or as a visually moderated sequence may produce the result that  $x=-1/2$  or  $x=1/4$ . Even if the result  $\sin x=-1/2$  or  $\sin x=1/4$  are found then a trigonometry frame which gives the implications of this result for the value of  $x$  will depend on a visualisation of the sine function moved from long term memory to the working memory, for the solution can not be seen in terms of a previously memorised result. Again, if the trigonometry frame is inadequate the solution can not be found.

When examining the quadratic prototype  $8x^2 + 2x-1=0$  the subject has to analyse the 'pattern' to select an appropriate strategy for the solution, again calling into account sub-procedures of factorisation. This bottom up approach may be seen as a 'means-end' approach, that is the subject has a clear view of the end to be reached and is searching for an appropriate procedure, or means, to achieve these ends. Noting the results quoted earlier about the resistance to recognition of variations on well learnt prototypes, it may be the case that students who have met quadratic equations predominately in the written form  $ax^2+bx+c=0$ , would not even recognise the original problem as being a distortion of the learnt prototype.

In mathematics, questions such as these the subject can normally assume that one of the strategies (s)he has met

will enable the solution to be found, so rather than a daunting means-end analysis of large numbers of possible options being required, a trial and error strategy of previously learnt procedures will often lead to solution.

This notion of problem solving is closely linked to Greeno's (1973) memory model, where it is postulated that the problem is constructed as a cognitive network in working memory. The next stage in the process is the construction of a network of connections between the variables of the original problem and the desired features which constitute a solution, and this process is carried out by means of using information from the semantic memory to modify the structure in working memory. Greeno describes the retrieved information as likely to be either 'rules' or 'relational information.' In Davis's terms Greeno is describing procedures and frames.

However, there is no real reference in Greeno's model of how the subject selects appropriate frames or procedures from long term memory in order to make the necessary connections. It seems likely that a scanning technique of possible frames takes place, which would mean that the likelihood of solving the problem is greatly increased if the problems themselves are limited to familiar types. In much of the traditional mathematics teaching children meet 'problems' after specific examples of similar types of questions, so the scanning of frames or procedures is limited to those most recently learnt.



There may be opportunities in mathematics for more open ended activities, where the subject applies reasoning skills from a repertoire of transformations which are recognised as mathematical activities, such as deductive reasoning in algebraic activities. The results may be linked to existing frames so to reflect on their meaningfulness. This kind of activity may be a major part of 'discovery learning'. In open ended activities the learner does not have the desired features which constitute a solution, but will need to reflect on the results of the transformations at intervals in order to search for meaning. Thus a child may rearrange pieces of a tangram to produce a rectangle from the original parallelogram but will need to reflect on this result to bring in to working memory a frame which involves area before (s)he is able to conclude that the rectangle and the parallelogram have the same area. However, much of the teaching in schools is directed at the acquisition of new concepts and in the example above the teacher can aid the acquisition of the desired result by cueing the 'area' frame at the outset of the task. In many instances it will be desirable that the teacher takes this role, and in the research, based on the acquisition of trigonometric concepts, the teacher will be aware of this 'cueing' process in student activity.



## 2.6 Attention and interference

Two of the major factors which influence information processing are pertinent to this research. The first of these is the attention given to incoming stimuli by the subject, and the second, 'interference', is concerned with the ways in which new data influences, and is influenced by, data already held in long-term memory.

When the subject is faced with a vast amount of stimuli it becomes necessary to select which of these are important to specific tasks, and this process of selection requires 'mental effort'. It is known that the state of arousal of the subject influences the amount of attention given to selection of stimuli with increased arousal (starting from low level interest) increasing the power of the subject to select appropriate stimuli, until the arousal level becomes stressful, creating anxiety, which then results in a decrease in the capacity to process information from incoming data. To this extent, the teacher of mathematics who is frequently seen as offering judgements on the ability of the subject may find, that when giving verbal information to individuals, the state of anxiety being created actually decreases the capacity of the student to process the incoming verbal information. In the same way the pressure to 'observe' relationships from limited data whilst being observed or questioned, which is common to much class teaching, may result in a restriction of the ability to

process the information and conceptualise relationships. This will be offered as a further reason for adopting the research based on the use of computer graphics; quite apart from the possibility of presenting information in a form which allows the subject to process it, it may actually increase interest without increasing anxiety, thus increasing the capacity of the subject for selecting appropriate stimuli. The Atkinson Shiffrin model postulates the use of a control device in working memory which can select appropriate stimuli for processing, but it acknowledges the fact that the initial stages of processing may be automatic. The subject may direct attention away from some information and away from others by conscious control but the interconnection of nodes in long term memory may become activated when one element in the network is activated, and this may happen without conscious control by the subject. Further, the more often the inter-relationship is activated, the stronger is the likelihood of automatic processing. In this sense we can see that when carrying out well practised motor skills we may only need to give a small amount of attention to the activity and think about other things simultaneously.

This model reinforces the notion of 'procedures' mentioned earlier, that is, well practised routines which may be cued from input and carried out with only a relatively small amount of attention. It could be argued that the acquisition of these automatic procedures is a



desirable state, leading to reliable responses which do not require a great deal of mental effort. If, however, the procedures are not linked to a wider knowledge representation structure, then the mental effort is only reduced for very specific examples, with variations leading to problems which can not be solved at all. If this is seen as the 'end' of mathematical activities then the efforts to employ techniques for widening a frame in the knowledge representation structure may not arouse the interest in the learner which the teacher may have expected.

The way in which we are unable to retrieve information from long term memory (theories of forgetting) have resulted in two major contributory factors which account for the phenomenon. The first of these is that of decay, which indicates that there is a very rapid decrease in the ability to recall information in the first hour or so after the information is received but that this decay is much slower after the initial rapid decline. Indeed there is a very low level of decay after a week or so after the information is processed. The second factor is that of 'interference', that is, the effect of new information on information already held in long term memory. Research in this area (Underwood B.J, 1964) reveals that 'retroactive interference', that is old information being displaced by new information, increases when the new information to be processed is similar to the old. However, there is also the phenomenon of 'proactive inhibition' in which new information is difficult



to recall because the stimulus associated with this information actually brings forth a response associated with some previously well learnt information which is in some way similar. In mathematics we may find that children try to use procedures which are quite inappropriate, or even apparently nonsensical if they associate a cue for the topic under consideration with procedures well practised when learning a different but similar topic. This is a factor which was observed in responses to the tests and interviews given to the children used in this research.

### 3. Cognitive Theories in Mathematics Education

#### 3.1 Preamble

This chapter relates the theories in mathematical education, and knowledge of how the brain works to the information processing viewpoint discussed in the previous chapter. In particular, it will examine 'pattern detecting' and the nature of 'intuition' with respect to the ways in which mathematics can be learned and discuss the usage of such terms as 'understanding'.

#### 3.2 Piaget, Skemp and Ausubel

The work of Jean Piaget in seeking to explain intellectual development has undoubtedly been one of the most important components in the theoretical framework of mathematical education. The basis of his theory is that intellectual growth is governed by the two principles of adaptation and organisation. The first of these describes the way in which the mind seeks to respond more effectively to the demands of the environment and is a two part process consisting of 'assimilation' and 'accommodation'. Piaget believed that mental structures take in or assimilate external events and converts them into mental events or thoughts. If the mental structure in existence encounters external events which can not be assimilated into the mental structure as it presently

exists, some change in these structures, or 'accommodation' may be necessary in order to accept the new situation.

The second principle, that of 'organisation', refers to the belief that the mind is structured in increasingly complex and integrated ways. The simplest level is referred to as the 'scheme', which is simply a mental representation of some action that can be performed on an object. As the mind develops, these schemes become more integrated and coordinated.

For Piaget, knowledge is action; what is known about an object determines what actions can be performed on it. Knowledge would thus progress from being limited to motor actions in the very young child to become more interiorised as thought sequences. Crucial to Piagetian theory is the proposal that the intellect progresses through four major periods of development. Within each stage change is quantitative and linear, whereas changes from one stage to the other are major qualitative changes. These four stages are categorised in terms of the approximate ages at which they occur, though it is fair to say that these 'ages' are thought by many critics to be dubious. The first of these is the sensory motor period (from birth to around two years), the second is the pre-operational period (until around seven years), the third is the concrete operational period (until around eleven years) and the final stage is the formal operational period (from around eleven to adulthood).



The children involved in this study, aged fourteen to fifteen years of age, would be either in the concrete operational stage, though rather late in developing the formal attributes, or indeed have progressed to the formal operational stage, so it is important to have an overview of the kinds of thinking associated with these two stages in Piaget's theory.

Children in the concrete operational stage will have progressed in their thinking, according to Piaget, in three areas: conservation, classification and in the understanding of transitivity. Piaget's experiments on conservation, particularly of volume, are well known and appear to show that children who have not reached the concrete stage are unable to appreciate an underlying invariance when some dimensions change. When a child understands conservation he or she can ignore perceptual changes and understand the reality of continuity and quantity. Before this stage the child can predict what will happen when, for instance, water is poured into a different shaped container and predict that changes in one dimension may cause changes in another, but he or she cannot integrate and coordinate these two pieces of information into a single system in order to appreciate conservation of quantity. The child in the concrete operational stage can interiorise the operations involved without actually carrying them out and can mentally reverse the operation to return the quantity to its original state.

The second property of concrete operational thinking is the capacity for classifying groups of objects into classes and subclasses. The concrete operational thinker can combine classes to make a third class and break the third class back into the two sub classes.

The third ability is that of being able to string together a series of elements according to some underlying relation and be able to conclude that if A relates to B and B relates to C then A must relate to C, the principle of transitivity. Until this stage the child may be able to observe that A related to B and B relates to C but be unable to coordinate these two facts to conclude that A relates to C. This particular property is essential to algebraic thinking so before this stage the deductive reasoning associated with much in algebra would fail to be appreciated.

The differences between the concrete operational thinker and the formal operational thinker is that the former is limited to the coordination of concrete things. though they could be represented mentally, in an actual situation and is limited to dealing with one system or dimension (such as number, volume or weight) at a time. The formal operational stage represents a development whereupon the child can integrate and coordinate previously isolated concrete systems. He or she is not dependent on the immediate concrete reality for dealing with problems but can think about them in a more systematic, formal manner. The formal



operational student, with integrated and coordinated systems can hypothesise and think at an abstract level rather than being tied to concrete activities.

Perhaps the first point to be made about the progression from concrete operational to formal operational thinking, as described above, is that there is little evidence of more than a minority of children being able to develop these attributes by the age of sixteen. The number of sixteen year olds able to work successfully with the abstract generalisations involved in the algebra associated with the G.C.E. 'O' level courses may lead one to believe that only around 25% are capable of operating at this formal stage. Certainly this would concur with the conclusion of Ausubel et al (1968) that only 13.2% of high school students and 22% of college students in the U.S.A. were at this level of development. More importantly, from the information processing viewpoint, Piaget's theory fails to take into account the range of processes described in the previous section, which are activated when the learner encounters stimuli. It is unlikely that all aspects of information processing activity develop at the same rate regardless of the kind of previous experience. Attention, for instance may develop separately from the capacity to visualise, or the skill of 'chunking' information in short term memory. Nor can the logical development described by Piaget in terms of seriation and transitivity be separated from the perception,



or pattern formation associated with the problem. Two examples show the difference between the two views.

Trabasso (1977) has shown that rather than apply logical deduction to form the transitivity relation for pairs of lengths, that is observing A as bigger than B then B as bigger than C then deducing A bigger than C, the children he tested actually constructed a mental image of the whole array and recognised the bigger difference between A and C more quickly, when asked to respond to the pairs than they responded to the original observed pairs.

Nicholson and Lucas (1984) describe the work of Maggie Mills of Bedford College, London University, as she worked with very young children on conservation tasks. These children, at the beginning of their schooling, would not be expected, in Piagetian theory, to have acquired the ability to understand the conservation of volume as liquid is poured into different shaped containers. However, Mills shows that when the language used is familiar to children and captures their attention, in this case describing the problem in terms of fairness as water is given to different animals requiring long or short containers, the children are quite capable of appreciating the conservation property. In this case it is the attention given and meaningfulness of the task which governed the thinking process.

Piaget's theory also appears deficient, in information processing terms, for failing to describe the processes which improve with age thus accounting for the development

he describes. One such factor could be the differences in the way children encode information. There is considerable evidence that older children are able to encode information in terms of appearance, sound and semantically, whereas younger children may be limited to only one or two of these dimensions. This capacity for multiple encoding would mean that the teacher of older children can present material in ways which allow for this multiplicity of encoding. Similarly, the way children are able to organise information into categories, (described earlier as 'chunking') will be a factor in the way in which they are able to use the information in higher order cognition. Indeed, Sternberg (1985) describes the growth of meta-cognitive skills, the mechanisms which control and organise others, as a major factor in children's development. This growth in the 'executive' function would supplement the growth in the capacity for multiple encoding in the sense that the coding would not only be available but be readily organised and selected.

Clearly the information processing view, which views the mind as a diverse collection of individual processes which do not necessarily follow the same rules contrasts with Piagetian theory. Piaget's theory is based on the assumption that cognitive development can be described in terms of a single set of principles. The information processing view provides a theory which appears more sensitive to the diversity of human thinking and abilities and the empirical



basis for arriving at the theories described earlier is persuasive. However, in terms of higher order cognition there is much in Piaget's theory which appears to complement the information processing approach. In particular, there appears to be no adequate description for the modification of knowledge representation structures as new information is perceived in information processing models. Piaget's principals of accommodation and assimilation appear to represent a coherent model of this development. The view that knowledge is action, what we know determines the actions which can be performed on an object, and these actions become interiorised as we develop, also appears to be a useful addition to the information processing model, describing the dynamic interaction between operations and knowledge.

Other important additions to the theory come from the work of Richard Skemp, particularly in his clear thinking on the development of concepts and his analysis of the term 'understanding'.

Skemp (1976, 1979), made a valuable contribution to the discussion on the nature of learning mathematics by making a distinction between two types of understanding, which has since become well established in mathematical educational terminology. He suggested that understanding could be seen as either 'instrumental', where the task is simply recognised as one of a class for which a rule for completing the task is already known, or 'relational', which relates



task to an appropriate set of concepts. The advantages and disadvantages of teaching to achieve these two kinds of understanding have been discussed by Skemp (1976), though it is almost self-evident that when there is conflict between teacher and pupil about which of these is being aimed for, the result is likely to be a lack of confidence in the mind of the pupil about the usefulness of the mathematics lessons. By contrast, if teacher and pupil are both aiming at the same kind of understanding, and pupils perceive that the lessons are contributing to its achievement then the lessons are likely to be received as 'good' mathematics teaching. There has been much debate about extending Skemp's simple dichotomy with, amongst others, Buxton (1978) describing 'formal understanding', Backhouse (1978) describing symbolic understanding and Haylock (1984), in a similar vein to Greeno's model, describing a consideration of understanding as connecting new and existing knowledge. However, for many the most important distinction is that between relational and instrumental understanding.

It may be seen that there is a connection between procedural thinking and instrumental understanding, though this connection may not be as clear cut as at first it appears. It is possible that children can learn mathematics as a series of disconnected procedures, each cued by appropriate stimuli, without recourse to a wider series of frames in the knowledge representation structure, and this would certainly represent instrumental understanding.

However, it is possible to have procedures which link concepts and these concepts may represent a relational understanding of the topic, even though the procedures themselves may not be understood relationally.

Pirie (1988) describes the difficulty of trying to assess the kind of understanding children may possess as they work with fractions, concluding;

"Much work needs still to be done by those who would catch and assess a child's potentially nebulous and fluctuating mathematical understanding" (p6)

In earlier writing Skemp (1971) uses 'understanding' in the sense of 'relational understanding' and is clearly Piagetian in his thinking, referring to assimilation of new concepts into existing schema. He also develops Piaget's view of accommodation, referring to the need to reorganise ideas when faced with "cognitive conflict", where new ideas conflict with an established cognitive structure. In this way Skemp describes concisely the need to adjust knowledge representation structures when incoming data conflicts with the parts of the cued frames. This is a useful addition to an information processing approach, for it may be that information perceived by one method of input can appear to cause this kind of cognitive conflict when it is being accommodated into a conceptual structure based on some other form of representation. In particular, a visual geometric approach to trigonometry may present information which



appears to contradict a calculation carried out on information received by reading numbers and incorporating a well learned procedure.

From Skemp's work there emerge two fundamental principles for the teaching of mathematics if understanding, in the relational sense, is to be achieved. These are:

1. Concepts of a higher order than those which a person already has cannot be communicated to him by definition, but only by arranging for him to encounter a suitable collection of examples.
2. Since in mathematics these examples are almost invariably other concepts, it must be ensured that these are already formed in the mind of the learner. (Skemp R, 1971, p32)

There is considerable agreement between the writings of Skemp and the earlier work of David Ausubel (1968, 1971). Ausubel introduced the notion of meaningful learning through the student's active reception of the ideas expressed by exposition by the teacher. His differentiation between 'meaningful' and 'rote' learning is particularly relevant to Davis's description of procedures and the Craik and Lockhart model of levels of processing. In referring to a hierarchically organised cognitive structure he says:

As new material enters the cognitive field it interacts with and is appropriately subsumed under a relevant and more inclusive conceptual scheme. The very fact that it can be subsumed (i.e. related to established elements of the present cognitive structure) accounts for its meaningfulness and permits the perception of insightful relationships. If it were not subsumable, it would constitute rote material



and form discrete and isolated traces. (1971, p340)

Ausubel distinguishes between the reception/discovery dimension, by which students encounter new ideas, and the meaningful/rote dimension which describes the way in which the student assimilates these ideas, establishing the fact that exposition by the teacher leading to reception by the student can in fact lead to meaningful learning.

Central to Ausubel's view of how meaningful material can be subsumed is the introduction of an 'advance organiser'. This would provide one or more anchoring foci for the reception of new material. He argues that it is unlikely that students have relevant subsuming concepts in their cognitive structure which could act as anchorage for new concepts, in which case it may be desirable to introduce appropriate subsumers and make them part of the cognitive structure before the learning task is introduced.

Ausubel describes a role for exposition by the teacher which accords completely with an information processing model but at the same time allows for the Piagetian model for accommodating and assimilating new concepts. His description of the 'advance organiser' provides an excellent model for linking visualisation with the more sequential deductive reasoning associated with formal algebraic thinking. Tall (1985) was to develop this idea further by introducing the 'generic organiser', in which the learner's attention is drawn to certain aspects of examples from which

the more abstract concepts can be meaningfully acquired. The teacher, in Tall's view, would act as an 'organising agent', focussing the pupil on the important ideas but encouraging pupil interaction with software designed to provide specific examples, in order that the student can make personal constructs of the inter-relationship of mathematical ideas. Ausubel's advance organiser, and Tall's generic organiser are both key factors in considering the role of the teacher in an information processing model of teaching.

### 3.3 The human brain and meta-cognition.

Evidence from physiological studies of the brain suggest that the Atkinson Shiffrin model of information processing can be matched to specific areas of brain activity. In particular, areas of the neocortex can be identified with spatial perception, auditory association areas, language abilities and, at the frontal lobe, an area which corresponds to the 'executive' selector in short term memory, associated with planning and organisation of action and thinking. In addition the more routinised action requiring less attention are associated with the evolutionary older sections of the brain, including the mid brain and brainstem. Complex representations are created in various parts of the neocortex, depending on the sensory nature of the information to be processed, with considerable fore-brain activity, in the decision making process, but as



skills become more automatic the demands on attention are reduced and the become represented by a different 'brainstem' system. In addition it can be established that perceptual processing operates largely as parallel systems, with separate regions of the brain processing aspects of perception which converge as the information is combined into a representation.

The lateralisation of functions of the cerebral hemispheres distinguishes the human brain from the brain of other primates. The left hemisphere, in most people, is specialised in processing complex sequences of information and in controlling complex sequences of movement, whereas the right hemisphere, in most people, is superior at simultaneously processing information from a variety of sources. The left hemisphere is thus associated with the control of verbal behaviour and reading, whereas the right hemisphere plays a pre-eminent role in visuospatial processing, aspects of emotional behaviour and the perception of certain musical forms. However, the distinction is not absolute; some verbal capacities are controlled by the right hemisphere. The right hemisphere appears to have poor syntactic and phonetic abilities and is best at comprehending concrete words associated with images (Lloyd P et al, 1984), thus supporting Paivio's suggestion that multiple encoding of words and pictures is most effective at the simple, concrete, level. However, as will be described in following paragraphs, it would be over



simplistic to view cognition in terms of isolated right or left brain activity.

The fact that different functions have been associated with the two hemispheres has led to the view that there may be two distinct learning strategies associated with the two sets of hemisphere activities; analytical/serialistic with left hemisphere and global/holistic with the right hemisphere. Brumby (1982) suggests the characteristics of analytical/serialistic learning are:

Immediately breaking a problem or task into its component parts and studying them step by step, as discrete entities, in isolation from each other and their surroundings.

The characteristics she associates with global/holistic learning are:

An overall view, or seeing the topic/task as a whole, integrating and relating its various subcomponents, and seeing them in the context of their surroundings. (p244)

The results of Brumby's study of the way students attempted to solve problems in Biology indicated that there were three distinct groups: those who used only global/holistic strategies, those who used serialistic strategies and those who used a combination of both. Those able to use both were described as "versatile learners". In the early stages of secondary mathematics the serialistic style of much of mathematical instruction may benefit serialistic thinkers. However, at the higher levels of mathematics, the versatile

learner , who can move between a local view and a global view will be advantaged.

In a study particularly pertinent to this research, Dreyfus and Eisenberg (1987) examined the effects of working with a computer program designed to encourage students to develop a global understanding of transformation of functions. The students, in the 11th and 12th grades, were encouraged to link this global approach with the analytical understanding to the algebraic representation of the functions. The study confirmed "the potential of micro worlds for promoting abstraction" (p190), but showed that students found it difficult to act as versatile learners. They report that "integration between the visual and analytic mode was achieved by only three out of the eight students" (p190). However, Blackett (1987), in research on the way in which children understand linear and locally linear functions using computer graphics, found that children could be encouraged to integrate the two approaches when the teaching method was more in the mode described by Tall as using the computer as a "generic organiser".

Gershen Rosen(1987) advocates a teaching style which recognises that pupils may respond to a global, visual approach to mathematics as well as the more serialistic approach. She says that:

....both left and right brain approaches are possible with most topics in mathematics and it is important for the teacher to at least be aware of this(p136)



There certainly appears to be differences in the way mathematicians prefer to think, with some thinking in pictures, as opposed to analysis of serialistically stored semantically code information. Ian Stewart (1989) discussing the work of mathematicians working at an advanced level is of the view that:

There seem to be two main types of mathematicians. Most work in terms of visual images and mental pictures; a minority thinks in formulas. Which type of thinking is used doesn't always depend on the subject-matter. There are algebraists and logicians who think in pictures, and I know that one leading topologist has real trouble visualising three-dimensional objects. (p95)

Shepard's work on the second order isomoprphism between images in working memory and the objects themselves makes clear that the way in which information is perceived is closely linked to the way in which the information can be used in thinking. The perception of data may depend on the attention being drawn to particular features, to allow the processing of particular pattern forms, rather than a serialistic processing of semantic features.

The Atkinson Shiffrin model accepts the need for some form of control or executive function to select the information to be used in short term memory, when the procedure is not automatic, and there is support for this in the research on patients suffering from lesions of the brain.



The ability to perform visually related tasks is particularly disrupted by right lesions in the neocortex in the temporoparietal region, whereas thinking about verbal analogies is most disrupted by left temporoparietal lesions. Lesions in the frontal association cortex appear to have very little effect on performance in conventional, short questions which may be require both verbal and visuospatial skills, though right or left temporoparietal lesions would badly affect these skills. Lloyd et al (1984), suggest that thinking depends on a number of specific processing capacities and that the questions associated with intelligence tests do not require " the orchestrating skills of the prefrontal cortex because they can be solved by using fairly automatic, stereotyped algorithmic procedures" (p124)

The authors go on to suggest that frontal lesions cause deficits in the ability to solve problems for which no obvious solution procedure exists, drawing on many of the temporoparietal cortex specialisms. There is then clear physiological support for the differentiation between visually cued procedures and more advanced problem solving skills. The 'word' problems in mathematics are likely to require these orchestrating skills in linking semantic information to mathematical frames and then to appropriate procedures and many children find these much more difficult than questions given in numerical or diagrammatical form which cue procedures directly.

Looking at the the problem of concept acquisition from the way the brain functions produces support for the information processing approach, with its emphasis on the diverse factors contributing to the ways in which we learn and the elements of theory outlined above, concentrating on the assimilation or accommodation of information into conceptual structures. Answering the question, "What does the brain, the organ for human learning, require for best learning to occur?" Leslie A Hart (1983) makes the following points:

1. The brain, the organ for human learning, took its shape tens of thousands of years before Greek-type, sequential logic was invented. *No part of the brain is naturally logical*

2. While a logical arrangement or presentation may serve the needs of transfer of information between people with good knowledge of the material, this kind of presentation produces consistently very poor learning results with students not already familiar with the material. Students prior to learning, cannot perceive the logic that may seem apparent to the instructor after learning. (p57)

He proposes a definition of the learning process as, "the extraction from confusion of meaningful patterns" (p67) and suggests that the brain is "by nature a magnificent pattern detecting apparatus" (p67). Hart advocates the use of many and variable inputs, with clues, to allow children to form meaningful patterns. The pattern recognition would depend on the experiences the children bring to the situation and



they must be allowed to revise these patterns to fit new experiences. Though this revision of patterns concurs with Skemp's principles of accommodation, and his emphasis on the experiences brought to a situation can be seen in terms of Ausubel's subsuming concepts, Hart makes much more of the probabilistic fashion in which the brain seeks patterns from the processing of such data given in many different forms. This view of the brain as designed to detect patterns from incoming stimuli can give further meaning to the notion of intuition and the terms used in mathematics education which amount to gaining a "feel for the subject".

Skemp (1971) defines functioning at the intuitive level in terms of being "...aware through our receptors (particularly vision and hearing) of data from the external environment; this data being automatically classified and related to other data" (p55). In information processing terms, Skemp is relating intuition to classification from pattern detecting in the process of perception. Perhaps a more recognisably general view of intuition, that is according more with the general use of the word and in particular the way it is used when psychologists refer to the right hemisphere as being "intuitive", would be Fischbein's definition as "direct, self-evident forms of knowledge". (Fischbein E., 1987). Fischbein's view of intuition as grasping the universality of a principle again describes the kind of activity referred to by Sperry (1973) when describing right hemisphere activity as holistic, non-



verbal, and intuitive and which is responsible for synthesising rather than analysing information. It is important to note that Fischbein does not limit intuitive thinking to visualisation, but maintains that it is an essential part of logical reasoning.

It is clear then that the belief in cognitive styles, concentrating on one hemisphere rather than another may be over simplistic a view. It appears more likely that the brain is adept at switching information from hemisphere to hemisphere across the corpus collosum, allowing intuitive, visual or holistic views to relate to the more analytical serialistic, verbal, processes associated with left-hemisphere thinking. Springer and Deutsch (1981) in a book often quoted by those advocating a more simplistic view of hemispheric processes, sum up their analysis of hemispheric activity as follows:

We question however the division of styles of thinking along hemispheric lines. It may very well be that in certain stages the formation of new ideas involves intuitive processes independent of analytical reasoning or verbal argument. Preliminary schemes ordering new data or reordering pre-existing knowledge could probably arise from aimless wanderings of the mind during which a connection is seen between a present and a past event or a remote analogy is established. But are these right hemisphere functions? We don't think it is as simple as that and there is no conclusive evidence to that effect. (p192)

It appears likely then that though there may be preferred kinds of thinking, thinking in pictures rather semantically

or vice versa, and that teaching can utilise both sets of activities, the brain is never limited to operating in one or other of the hemispheres but is capable of switching information between the two kinds of encoding and operations. The original assumptions about left and right hemisphere styles, heavily influenced by research with split brain patients, appears to greatly oversimplify the complex interactions between the various parts of the brain. This is certainly the view of Michael S. Gazzaniga (1985) one of the most prominent of researchers with split brain patients, who asserts that the brain works in a modular form, with specific brain systems handling specific tasks, but that these modules are inter-related. He does not accept the simplified view of Sperry and others of distinct right and left hemisphere activities, declaring that, "isolating mental systems or claiming that isolated mental systems process information differently does not really illuminate the nature of cognition" (p59). In order to overcome this problem Tall (1990) has adopted the phrase "metaphorical right brain" to refer to those activities which, in earlier work have been assumed to be isolated right hemisphere activities.

Fischbein introduces the idea of primary and secondary intuition, the first growing out of practical everyday experiences and according more with Skemp's view of operating at an intuitive level. This primary intuition grows as part of a cultural ontogenesis, independently of



systematic instruction, out of everyday experiences. By contrast, secondary intuitions have no natural roots but depend on the formal structure of the subject. In Fischbein's view there are frequent conflicts between the primary intuition, arising from our experiences in sensory interpretations and secondary intuitions depending on the formal structure of mathematics, an example being the acceptance of  $-a \cdot -b = ab$ , which can not be abstracted from an intuitive model. In other words, it is necessary for students to suspend their primary intuitions in certain settings in order to build the patterns necessary for secondary intuitions.

An intuitive model, according to Fischbein, is a substitute constructed to cope with a notion that is intuitively unacceptable and essentially sensory. They are cognitive constructions made by someone to make sense of a situation and should not be confused with instructional representations. He observes that the use of diagrams involves the establishment of a number of conventions that specify the meanings of the images used. Diagrams then belong to Bruner's symbolic mode and the intuitiveness of diagrams:

is not a natural property. One cannot rely directly on the respective images in order to get a more convincing grasp (in the intuitive sense) of the concepts and relationships expressed... *Diagrams are not, generally, the direct image of a certain reality.* (p157)



He goes on to emphasise this point:

A diagram, although expressed in figural terms, is not a primary cognitive instance. It is the figural expression of an already elaborated conceptual structure, like any other symbolic system. (p157)

This is an important distinction which plays a key part in the development of the teaching involved in this research. The pattern formation involved in primary intuition can involve an intuitive model, and computer graphics can provide this kind of intuitive model. At some stage there may well be the need to suspend some part of this intuitive primary intuition in order to build a secondary intuition, but the model itself, using visual stimuli and inviting the use of either geometric visualisation or numerical representation, is essentially different from the symbolic representations categorised by the more typical trigonometric diagram.

#### 4. The Research Thesis

The teaching of trigonometry for pupils between the ages of 11 and 16, is beset by difficulties, to such an extent that the G.C.S.E. now typically requires that only the intermediate and higher levels should study the subject at all. At the intermediate level the students would typically be required to recognise the trigonometric ratios of sine cosine and tangent and be able to solve problems using the right angled triangle. At the higher level students would be expected to use trigonometry to solve three dimensional problems restricted to right angled triangle problems.

The problems experienced by students studying trigonometry in the fourth and fifth years of secondary school are not difficult to appreciate. Understanding trigonometry, in a relational sense, depends on understanding of ratio, in the most abstract sense of similar figures, which has been found to be even harder for children than using ratio to share amounts. (Hart K. M., 1981,). In addition there is the introduction of new terminology (sine, cosine) which are essentially meaningless in terms of past experience and the term 'tangent' which appears to have no link with any previous usage of the term. The 'words' are often defined as ratios, such as 'sine= opposite/hypotenuse', a form which exists in the knowledge representation structure of most students only as a

description of how to calculate a quantity such as area ( $A=l \times b$ ). It is not surprising that students who can calculate 'sine' from this 'formula' may have some doubts that they are calculating some quantity which they can not identify,

A relational understanding of the ratios themselves depends upon the detection or intuition of two essential patterns which are themselves essentially geometric and dynamic. The first of these is that in any right angled triangle, when an acute angle is enlarged and the hypotenuse remains fixed, the opposite side to the angle being enlarged grows in size as the angle increases but the third side, the adjacent side, is reduced in size. There are singularities to be considered here, that is the limiting values of the acute angle at 0 and 90 degrees, at which point the students must suspend primary intuition (the triangle does not exist) and accept values for the resulting lines. The second pattern is the enlargement of the hypotenuse by any given factor results in an enlargement of the other two sides by the same factor. Because of the highly abstract nature of this understanding it would be necessary to build the higher order concept from examples based on a series of representations.

It is likely that for many children the basic concepts of 'triangle', 'acute angle' and the ability to estimate angles and sides will be underdeveloped before the topic is taught, so the ability to visualise is strictly limited and



children will select numerical options as a mode of working. The problem arises then of how to allow children to develop examples in such a quantity that they are able to appreciate these patterns. If a strategy of drawing examples is employed the student has to concentrate on the method of construction and is unlikely to produce enough examples. If, in addition, they are expecting a procedure to result from their activities, there is a danger that they will suspend their search for meaning until they receive a recognisable procedure which can be associated with success. Because the procedure is not meaningfully learnt, and, for the reasons stated above, trigonometry has an inherent potential for encouraging meaninglessness with the very words that are used to define the concepts, there is a danger that instrumental understanding is seen as the only kind of understanding available.

The procedures associated with trigonometry can be linked to visual cues and mnemonics, so the student searches for such visual cues, such as  $\text{sine} = \text{opp}/\text{hypotenuse}$ , and seizes them with alacrity as a means of cueing procedures. In addition, much of what pupils associate with successful mathematics, the ability to complete calculations, is heavily procedure based, even though the procedures may be understood relationally or be part of a wider frame. Given the reduction in mental effort associated with routinised procedures, it would be surprising if they did not find the

possibility of producing correct answers by reproducing procedures an attractive prospect.

These procedures often involve around three statements with an apparent algebraic link so that the complete procedure would be something like:

$$\sin 30 = a/3$$

$$3 \sin 30 = a$$

$$3 \times 0.5 = a$$

$$a = 1.5$$

At line three in the procedure the student needs to find  $\sin 30$  using a sub-procedure on a calculator or book of tables.

Given enough of these procedures which are visually cued from a 'question', it is possible that students can appear to demonstrate a relational understanding of trigonometry. However, there are obvious drawbacks to this, the main one being the natural decay in long term memory of the procedure itself, which can cause efforts to reproduce it to result in visually similar statements which are mathematical nonsense, such as  $\sin 5\text{cm} = 30/10$ . Wider knowledge representation structures, involving for instance the nature of the limitation on angle size in a right angled triangle are not cued by the incoming stimuli, so it may be possible to work through the procedure and arrive at an angle which a visual reconstruction of the triangle would reject as impossible. By the same token, if all that is asked of students is a reproduction of visually cued procedures it may be

impossible to tell whether students who had worked through the procedure correctly had any relational understanding of trigonometry, or indeed whether they had any sense of angle size and a meaningful interpretation of the result.

As the student moves from the sine of the angle to cosine and tangent, it is frequently the case that some form of mnemonic is introduced which is a verbal or visual cue to the appropriate procedure. This mnemonic can be rehearsed until it is well established in long term memory, but it inevitably involves the identification of sides of the triangle as 'opposite', 'adjacent' and 'hypotenuse' and therein lies a further problem, particularly if the triangle in question is presented in a different form to a well learnt prototype. The prototype is frequently given with one side horizontal and one vertical, so any rotation of this can cause the student to fail to identify the sides, and may even fail to cue the trigonometry frame at all.

Whether the ratios are understood or not, the questions the student associates with trigonometry are not likely to involve the multiple encoding of concepts. The diagrams are, in Fischbein's terms, symbolic, and are scanned by the learner and analysed by selecting the appropriate features for substitution into the procedure. This point is often emphasised in questions, so that the learner is aware that the solution is to be found by "calculation", that is the diagram can not be taken to be a representative model but a



shorthand symbolic way of presenting data for entry into the procedure.

Many of the mathematics text books in use in secondary schools, including those used by pupils in this research in earlier work, seek to avoid the inherent difficulty of the abstract nature of the ratios by defining the sine and cosine of an angle as the opposite and adjacent sides of a right angled triangle with a unit hypotenuse. In this way there is a concrete entity which can be labelled as the sine or cosine of the angle, and in Piagetian terms the abstract nature of the trigonometrical ratios can be built from the operations on the concrete systems. Unfortunately there is still the difficulty of the two essential dynamic patterns, that of enlarging the triangle to show that as the hypotenuse increases by a factor of  $x$ , then the sine and cosine increase by the same factor, and the pattern of changing values of sine and cosine as the acute angle varies in size.

It is difficult for the learner to concentrate on these global patterns when constructing their own triangles, as (s)he is involved in a number of serialistic activities, such as measuring angles and drawing lines of given lengths, and text books have great difficulty in indicating these changes from static diagrams. As a result, it is possible that the children presented with this kind of diagram and expected to see the relationship expressed by the similarity fail to do so. If they then do not have the fall back of a

procedure to use it is likely then that they would be left feeling completely bereft of ideas. Children in this state would express their utter confusion with trigonometry, whereas those who have learned the procedures, however instrumentally, feel that they can answer the questions expected of them. In the experience of the writer, it is quite common for students who have followed the newer text book introduction to trigonometry, based on the unit hypotenuse, to feel a great sense of relief when parents or tutors give them the procedures and mnemonic associated with the more traditional method. This belief, that they only 'understand' because someone outside of school has shown them a procedure was common to many of the students who responded to the pre-test in this research.

This research argues that the definition of sine, cosine and tangent of an angle, based on the unit hypotenuse or adjacent side, can be made meaningful for students if incorporated into a computer graphics program which is essentially dynamic and incorporates the dual encoding of visualisation and numerical notation. The pattern formation would be based on a scaled accurate representation of a triangle which could be enlarged or rotated or both, and which would be instantly redrawn if variables are changed. The input of variables, hypotenuse, opposite side, adjacent side or acute angles, would be under the control of the user as would the rotation or enlargement, and the numerical



value of the other variables would be available on a different section of the screen.

In this way, the essential nature of the trigonometric patterns would be recognised from the perception of the many visual images which could be drawn at a speed, allowing the student to focus on the change in particular variables. Perceptions of sizes from the computer images can be compared with actual numerical values. In the same way, when calculations are made using the procedures associated with statements such as  $a = h \sin x$ , where  $a$  is a length of side to be calculated,  $h$  is the value of the hypotenuse and  $x$  is the size of a given angle, the result could be checked by producing the triangle in visual mode and comparing the calculated result with the visual image and the value given numerically on the screen.

The closeness of the visual imagery to the perception of visual stimuli, in Shepard's term a "second degree isomorphism" is such that students would be encouraged to make visual images of the perceived information. In this way the visualisation capability of right hemisphere activity would be stimulated. Students would be encouraged to select appropriate image forms in working memory, with visualisation based on the perceived patterns generated by the computer graphics. Whilst the 'metaphorical right-brain' would be more active than in the more usual serial representation of trigonometry, there would be considerable movement between the numerical and visual form. The



organising executive function associated with the Atkinson Shiffrin model and related to the activities of the frontal lobes of the neo-cortex would be active in selecting appropriate modes and organising switches of representation.

The mental calculation involved in comparing the unit length with the length of the sides to be estimated allows for the restricted size of short term memory; the image is retained on the screen and the numerical value is revealed on another part of the screen. The user is thus moving from perception to short term working memory to long term memory, and checks are made by the same series of processes incorporating the numerical encoding.

Use of procedures and mnemonics would be encouraged but they would be placed in a wider, conceptual, knowledge representation structure, calling upon frames involving knowledge of angle sizes and the ability to compare lengths. In this way it could be assumed that students could link the procedures to a wider frame of knowledge without any detrimental effect occurring from the lack of practice of the procedures compared with teaching which establishes such procedures at an earlier stage.

By encouraging this versatility of representations, the student would be moving toward the situation where the essentially symbolic diagram, perceived visually from other sources than the computer drawn figures, can be mentally reconstructed to give a more accurate visual representation. This representation can be used for the estimation of the

lengths of sides or value of missing angles, before or after the calculations are made. In the same way auditory received information could be held in short term memory and transferred into visual representation.

The role of the teacher in the classroom where this approach is being used would be to direct the attention of the learner to the visualisation and numerical representations and to design activities which generate the patterns associated with the concepts to be acquired. (concepts in the wider sense, including Gagne's 'principles'). In addition the teacher could be expected to introduce meaningful verbal learning in the sense of presenting advanced organisers for the range of activities and consolidation of the concepts acquired by reviewing the activities and providing opportunities for practising written procedures.

Questions of accommodating cognitive conflict would occur when dealing with the singularities of sine, cosine and tangent of 0 and 90 degrees. This would be faced by students being taught by any method and requires the suspension of primary intuition, (the triangles don't exist), to allow the secondary intuition associated with trigonometry as part of a mathematical system. Students who had acquired this secondary intuition by following the computer based dual encoding approach would reject values of sine or cosine greater than one unit but would accept a value of sine or cosine as 0, those students who had only



acquired a facility with procedures may accept a value of sine or cosine greater than one but intuitively reject a value of 0.

The research seeks to establish the veracity of this theory by designing a series of lessons for a wide range of ability incorporating the computer graphic based, dual encoding approach to be taught to four experimental groups which have been matched to four control groups. The control groups would be taught by experienced teachers using a combination of worksheets, text-books and black-board work. The effectiveness of the teaching would be measured by applying two post-tests, one immediately following the lessons and the second after a delay of approximately six weeks, and a series of interviews with members of control and experimental groups.

The tests will be designed to assess whether students in the experimental group, taught using the computer graphics dual coding approach, have a better conceptual understanding than students in the control group and whether the experimental groups can apply the procedures necessary to solve typical trigonometry questions as efficiently as students in the control groups.

Both sections of both tests will be analysed using one way analysis of variance with the null hypothesis being that there will be no difference in performance between the two groups, and tested at a significance level of 0.05, using a two tailed test. A non-parametric test (the Wilcoxon signed



rank test) using matched pairs, tested at the same level of significance will also be applied. The research design will be described in detail in chapter six.

A major issue raised by this research is the possible effects of the experimental treatment on the performance of girls and boys considered as separate samples. The following chapter discusses the performances of boys and girls in mathematics and the possible effects of teaching the experimental groups in the way described. The tests will be analysed with respect to boys and girls separately, with two more null hypotheses: there will be no difference in performance between boys in the experimental groups and those in the control groups and no difference in performance between the girls in the experimental groups and those in the control groups.

## 5. Possible Differences in Performance Between the Sexes.

### 5.1. Preamble.

There is some evidence that male and female differ in mathematical attainment in adolescence and this research is an opportunity to investigate this issue. In particular, the emphasis on different perceptual skills and the movement between thinking in visualisations of geometrical representations and numerical forms is apposite to the general debate on differences in performance between males and females on standard tests of spatial abilities and other factors in cognition.

The chapter will move from an overview of evidence for differences in performance to an examination of the evidence for physical differences in brain structure and cognition, before discussing the relevance of the evidence to this research.

### 5.2. Differences in attainment in mathematics.

The difference in performance in mathematics between boys and girls can be seen from the statistics on G.C.E. 'O' Level and C.S.E. results for 1985, (D.E.S., 1989) and the D.E.S. 10% sample of school leavers for 1985 and 1987 (D.E.S., 1989) as well as from the Cockcroft Report appendix (1982) and the results of the Assessment of

Performance Unit Secondary Survey Reports (A.P.U., 1980, 1981). In addition results from national mathematics assessment in the U.S.A. (Lindquist M.M. ed, 1989), carried out for the National Council of Teachers of Mathematics, show similar findings to the A.P.U. results with respect to differences in performance between boys and girls at various stages in their schooling.

The G.C.E. results for 1985 reveal that more boys than girls entered G.C.E. mathematics and that they achieved better results overall, with 52% of the entry being boys and 60% of the boys gaining the higher grades of A,B, or C. By contrast, only 52% of the girls attained these higher grades, so that overall the girls accounted for only 44% of the total number of grades A,B, or C. The D.E.S. 10% leavers survey confirmed this difference for 1985 and showed that the results were virtually unchanged in 1987, highlighting the differences between boys and girls at the extremes of the qualification ranges. The results can be summarised by examining the number of boys and girls achieving grades A and B and the numbers obtaining grades 4 and 5 in the C.S.E. examinations.

	1985		1987	
	Grade A	Grade B	Grade A	Grade B
Boys	7.6%	10.1%	7.3%	9.5%
Girls	4.0%	7.9%	4.3%	7.6%



	1985		1987	
	C.S.E.4	C.S.E.5	C.S.E 4	C.S.E.5
Boys	12.8%	7.7%	13.2%	7.2%
Girls	15.4%	9.8%	15.3%	9.3%

(Source D.E.S., 1989)

The Cockcroft report (1982) shows that these figures vary very little from the results of 1979.

The Assessment of Performance Unit secondary survey (1981) highlight the differences for the most able by commenting that the differences in performance between boys and girls were most marked for those pupils whose scores fell in the top 20% of the ability range. Similarly, the investigation into mathematical attainment carried out as a national survey in the U.S.A. (Lindquist M.M., 1989) discovered that there were significant differences between the performances of boys and girls at the age of seventeen, in favour of boys, at the higher proficiency levels but not at the lower proficiency levels. The report states that the finding, "suggests that the gender differences in mathematics achievement result from the best males performing at higher levels than the best females" (p152)

The difference in performance in mathematics at the higher end of the ability range at the age of 16 becomes further exaggerated when the element of choice between subjects is enhanced in the sixth-form. In the academic year 1985, there were similar numbers of boys and girls

taking one or more G.C.E. advanced level courses, but 51% of the boys chose to study mathematics at A level compared to only 25% of the girls. The pass rate for boys and girls was however almost identical. There is clear evidence here to support the A.P.U. conclusion that girls tend to over-rate the difficulty of mathematics, that is they are able to answer more questions correctly than they are prepared to say are "easy". This concurs with Leder's conclusions (Leder G., 1980) that bright girls tend to display a fear of success response in Mathematics in far greater numbers than boys.

The advent of the G.C.S.E examination has brought some new teaching approaches in mathematics, with perhaps a greater emphasis on discussion and investigational work. The Royal Society publication, 'Girls into Mathematics' (1986) put forward theories on why girls underachieve in mathematics and recommended courses of action for various sections of the educational community. Classroom teachers were advised to have the same expectations for girls as boys and to give praise accordingly, but were also encouraged to involve pupils in discussion, to bring a social element to teaching and to introduce group work and cooperative teaching styles. It also pointed out that boys should not be allowed to dominate such discussion groups. It might be expected then that there would be some improvement in girls performance in the newer examination, where such activities have been actively encouraged

compared, to the more traditionally taught G.C.E. 'O' level examination.

The analysis of one examination board's G.C.S.E. results for 1988 (D.E.S., 1989) can provide a comparison with the results obtained in the two D.E.S 10% leavers' surveys of 1985 and 1987. When the two leavers' surveys are adjusted to give the numbers obtaining each grade as a percentage of the entry for G.C.E. and C.S.E., rather than as a percentage of the number of leavers, they reveal that the percentage of each entry obtaining the highest grade (A) was as follows:

	1985	1987
Boys	9.4%	9.1%
Girls	4.8%	5.2%
Difference	4.6%	3.9%

For the single examination board result for 1988 the results show that 7.4% of the boys entered obtained an A and 4.8% of the girls, a difference of only 2.6%. It could be inferred then that there had indeed been some improvement in the performance of girls at the highest end of the ability range. However, these results indicate that the narrowing of the difference is not accounted for by an increase in the performance of the most able girls but by a relative reduction in the performance of the most able



boys. A similar comparison for the higher grades, A, B, and C gives the following results:

	1985	1987
Boys	41.2%	40.5%
Girls	33.7%	33.5%
Difference	7.5%	7.0%

The single board G.C.S.E. results for 1988 show that 40.2% of boys and 33.1% of girls entered, obtained the higher grades. The difference remains stubbornly at around 7% of the entry, despite the change in the examination. (Source D.E.S., 1989)

Of course the G.C.S.E. was in its very early stage in 1988 and changes in teaching style take some time to become established. Nevertheless, there is little encouragement from these figures to suggest that the performance of girls, relative to boys, in external examinations, has been radically changed since the 1979 figures, quoted in Mathematics Counts (1982). A new analysis of G.C.S.E results by Helen Patrick, research officer for the Cambridge university local examination syndicate, appears to confirm that the G.C.S.E. examination has not affected the differences in performance between boys and girls. In her research, described in "The Times" (1.9.90), she asserts that investigations into differences in performance between the

sexes in two years G.C.S.E. results shows that the new examination has not greatly affected the differences in performances between boys and girls. Amongst other results it appears that girls outscored boys in English whilst "boys did consistently better in mathematics" (The Times, 1.9.90), and these differences were common to all examining groups.

The G.C.E. and C.S.E. examinations questioned a wide variety of mathematical concepts, as does the G.C.S.E.. It should be considered whether boys and girls differ in their ability to answer questions on some of these concepts more markedly than others. Two pieces of research illuminate this issue: the first was influential in attempting to consider the likely outcome of this research as far as male and female performance is concerned and the second gave a later view of the issue, but was published after the research had been carried out.

Wood (Wood R., 1976) looked at the G.C.E. examination papers of 493 boys and 478 girls who sat the examination in 1973 and a further 507 boys and 406 girls who sat the examination in 1974. He was able to examine the success rate of boys and girls in general for the topics being tested but was also able to compare boys and girls from the same sample schools in order to assess whether the teaching itself played a part in the differences in performance.

Wood concluded that boys had a far superior performance in questions involving scaling and ratio as well as for locus questions. The biggest difference of all in terms of the percentage of boys and girls able to provide the correct answer, 42% in favour of the boys, occurring for a question involving a three dimensional problem. He characterised much of the difference in performance by the superiority of boys in spatial skills, though it may be debatable whether the skills needed to answer some of the questions quoted by Wood really do depend on spatial awareness. Some, such as a question involving finding the number of children represented by a sector of a pie chart, where more girls than boys failed to convert the calculated angle for the sector back to a number of children, appear to be unconnected to spatial awareness at all. Perhaps Wood's observation that girls were more likely to complete only one stage of a problem gives a more meaningful explanation for this kind of error.

When trying to assess the effect of differences in teaching style on the success rate of boys and girls there is the added complication that boys and girls were taking other subjects which could contribute to their ability to answer mathematics questions. With respect to the three dimensional problem, in which the boys were so markedly more successful, for instance, Wood accepts that the question could have appeared in the same form in technical



drawing lessons which were not traditionally considered as a suitable option for girls.

When comparing boys and girls from the same schools Wood concludes that there was a great variation from school to school on the differences in performance between boys and girls on specific questions, most markedly in the ability to answer questions on distance/time graphs, linear scales and locus questions. He felt confident enough from the evidence he obtained to conclude that, "in some schools sex differences on certain problems can be reduced or made to disappear altogether" (p158)

It appears, then, that Wood's analysis would predict a difference in performance in trigonometry based on an awareness of scaling and rotation, rather than the establishment of procedures based on manipulation of equations, would favour boys rather than girls. If this were the case the experimental groups would show a greater difference in favour of boys than the control groups. However, the effects of different teaching styles were impossible for Wood to examine in detail. He was able to see that differences existed between schools and that these differences could well be accounted for by the ways in which the topics were taught, but he could not examine the teaching itself to see what the differences were. It will be argued that the computer graphics approach, integrating visuo-spatial and numerical codes will, in fact, allow girls to gain a conceptual understanding of

trigonometry more successfully than the more typical approaches to teaching this topic.

Bradberry (1989) followed up Wood's initial study by comparing the results of boys and girls who took the Northern Examining Association joint G.C.E. 'O' level and C.S.E. examination in 1986. He was thus able to cover a wider ability range than Wood, though the random selection of 500 girls' and 500 boys' scripts from the schools, mostly mixed comprehensive schools, from the north of England did not facilitate a comparison of the effects of the different schools.

Bradberry's results are remarkably similar to Wood's, with the multiple choice questions and the two other papers all giving consistent results on the topics in which differences in success between boys and girls occurred. In all three papers the boys performance was similarly statistically significantly better than the girls, so that the suggestion that boys have an advantage on multiple choice question format (Murphy, 1978) was shown not to be a major contributory factor accounting for the difference in performance.

Bradberry showed, consistent with Wood's findings, that the difference in performance was most marked for ratio questions, bearings, proportion and three dimensional work. The questions involving percentage and ratio revealed the inability of many candidates, particularly girls, to deal with different orders of units



and that this was compounded by the inability to relate the significance of answers to the situations that were being represented. He observed that girls did well on questions which required the use of standard procedures and this can be seen from the slight superiority the girls had in answering questions on the sine rule, cosine rule and a routine trigonometry question but having a much lower performance on other trigonometric questions where the question was more geometric. Again, the bearings question provided a major stumbling block for the girls, as did other questions which Bradberry associates with spatial visualisation, especially those involving scaling.

Bradberry, like Wood in the earlier research, is prepared to accept that questions which depend upon some form of spatial visualisation reveal some of the greatest discrepancies between the boys' and girls' results and that these together with the ability to deal with proportionality and scaling expose girls' weakness in mathematics.

On a larger scale, the first A.P.U. secondary survey (1980) reported the findings of written and practical tests given to approximately 10,000 fifteen year old pupils in England and about 2,500 each in Wales and Northern Ireland in 1979. The survey gave tests in clusters of topics and analysed the results in terms of sex differences and several other background variables. These results concur with the results of Wood and



Bradberry in that for every cluster of topics the boys' performance was superior to the girls' with statistically significant differences occurring in eleven of the fifteen item clusters. However, the greatest discrepancies in terms of differences between mean scores for the item clusters, occur for rate and ratio, (6% difference), mensuration (5% difference) and descriptive geometry (7% difference). The last topic concentrated on calculating values of angles in triangles and other polygons with one item on tangents to the same circle resulting in an isosceles triangle. Three of the five items tested in this item cluster depend on the ability to utilise the knowledge that the angles of a triangle add to 180 degrees.

There is a marked difference between these results and those obtained in the A.P.U. survey of eleven year olds (A.P.U., 1980a), where the boys' performance was superior to the girls' in ten of the thirteen sub-categories, but these were of the order of one or two per cent and girls' scored more highly than boys in the other three sub-categories with a four percent superiority in one of these, (whole numbers and decimals).

The A.P.U. statistics also confirm the analysis of G.C.E. results and the national U.S.A. survey detailed earlier, in that the higher mean scores obtained by boys were due more to the preponderance of boys in the top performance bands than to the preponderance of girls in

the lowest band. In the top 25% performance band the ratio of numbers of boys to girls was 58:42, but in the top 10% performance band the ratio of boys to girls was 61.5:38.5. For lowest 10% performance band the ratio of boys to girls was 48:52.

The trigonometry item cluster showed only a 3% mean difference in favour of the boys, though the mean scores for both were much lower than for any other item cluster. An examination of the test items, however, reveals that the items could have been answered correctly by learning procedures and reproducing them at the appropriate cue; no attempt was made to assess whether the pupils had an understanding of the similar triangles upon which the concepts are based or any ability to meaningfully interpret the results.

It would appear then that boys perform more successfully than girls in mathematics at the age of fifteen and that this superiority translates into superior results in external examinations. It would also appear that this superiority is more marked at the highest ability levels. The difference in performance appears to accompany an attitudinal difference, so that girls overrate the difficulty of questions in relation to their ability to answer them successfully, and their willingness to study the subject at an advanced level is markedly below what could be expected from results in examinations at sixteen. Within the overall pattern of performance the



evidence from A.P.U. tests and analysis of examination performance suggest that girls have more difficulty than boys in questions involving ratio, scaling and visuo-spatial questions involving angle and position. However, with respect to this research, questions on trigonometry, for which an understanding of ratio and scaling could be seen as essential, there is some evidence from both the analysis of examination questions and from the A.P.U. tests that the difference in performance is not as marked as for straightforward geometry or ratio questions. It could be assumed that for some questions, the learning of procedures, not necessarily linked to an understanding of ratio or scaling is sufficient to gain a correct answer. The following section is an examination of the research on psychological and physiological differences between boys and girls which may contribute to the differences in performance in the aspects of mathematics outlined above.

### **5.3 Perception and cognition: evidence of differences between the sexes.**

Large scale surveys and controlled testing procedures have produced some consistent findings on the differing perceptual and cognitive abilities between the sexes. Maccoby and Jacklin (1974) show that females aged one to five are more proficient in linguistic skills, showing an early aptitude for using speech purely for



communicative purposes. The verbal superiority is reduced in middle childhood, though females retain a slight advantage in fluency and comprehension. However, there is an outstanding superiority shown by girls in reading skills in early schooling with remedial reading classes containing a significantly higher number of boys than girls. (Ounsted and Taylor, 1972; Maccoby and Jacklin, 1974). In adolescence and adulthood females once more emerge as superior in verbal abilities. McGuinness (1976), suggests that early schooling places great emphasis on verbal skills, that this emphasis might improve the performance of boys, and that fixed age group teaching might handicap females, thus slowing their progress. However, she suggests that the natural ability of the female, though partially curbed, may re-emerge later when more complex verbal skills are necessary.

McGuinness is of the view that a good learning environment can reduce the differences in performance associated with superior natural abilities and evidence from Maccoby and Jacklin (1974) on the enhanced superiority of females in middle childhood amongst children from deprived families supports this opinion.

Female superiority has also been noted in most forms of visual and verbal memory tasks. Mittler and Ward (1970) report a female superiority in verbal recall amongst children and experiments by Paivio (1971) on the speed and accuracy of recall of items given in the form of

pictures, concrete nouns or abstract nouns, concluded that there was an observable female superiority. In these tests the female superiority was most marked when comparisons were made between male and female subjects classified as 'high visualisers' and for all subjects the superior performances were more likely to be in the recall of pictures. Paivio's research is highly relevant to this research, providing evidence that females are superior in recalling pictures when the stimulus was perceived in verbal form, and that the differences were more marked for female high visualisers over male high visualisers. The ability to move between visualisation and interpretation in numerical form is a key factor in the design of the computer based approach.

In terms of the way in which information is processed and responses to stimuli, there are noted differences between the sexes. When information is given in the form of a visual display of shapes, and the response requires large muscle units, such as operating a lever or moving a disc to coincide with ring, boys are superior to girls to an overwhelming degree, (Cook T. W., Shephard A.H., 1958) and this superiority carries on into adulthood. However, when the information is given in symbolic or semantic form and the response is required in fine motor movement, females are superior to males. (Tyler L., 1965). McGuinness (1976) believes that the distinction between visual figural input and semantic and symbolic input is



highly relevant in explaining sex differences, She also concludes that though such differences in processing are evident, girls in fact process information more quickly than boys and that available data supports the idea that girls are able to attend to a greater amount of information at any one time.

Perhaps the most marked difference between males and females occur in their performances in tests intended to measure spatial ability. Spatial ability has been variously defined but appears to consist of two major components, namely orientation and visualisation. Spatial orientation refers to the ability to comprehend the arrangement of elements within a visual stimulus pattern, the ability to deal with changing orientation and the ability to determine spatial orientation with respect to one's body. Spatial visualisation refers to the ability to mentally manipulate rotate or invert a visual stimulus. In terms of the information processing viewpoint described in chapter two, spatial ability refers to the perception of visual patterns and the ability to manipulate these in working memory. Shepard's work, described in chapter two, showed that the visualisation process is closely linked to the perceptual processes.

The ability to remember geometric shapes and then to find them when they are embedded in complicated geometric shapes, the embedded figures test, is more developed in older boys (12-18) than in girls of the same age, though



such differences are small if at all in children in the 5-10 age range ( Witkin et al, 1962). In the more kinetic tests, where mental manipulation of objects are required, such as counting the number of block surfaces visible from perspectives of stacks of blocks initially shown in two dimensional pictures from different viewpoints, boys show a high degree of superiority and this superiority is maintained into adulthood (Stafford, 1961). Similarly, the ability to trace the correct path, with a pencil, through a variety of mazes is very much superior in men. Again, these differences are much less evident in younger children, with test results showing that that 5-6 year old boys and girls were equally able to roll a ball through a maze (Harris L. J., 1978).

Witkin et al (1962) investigated the ability of males and females in setting a rod which is surrounded by a tilted frame to the true vertical, (the rod and frame test). He found that males performed much better in this task and that the differences appear earlier and more reliably than in the embedded figures test which is given at the same time. In the rod and frame test male superiority appears in children in the 5-10 age range and beyond this age the male superiority is routine.

The Piagetian task of drawing the water level in a picture of a tilted bottle would appear to be similar to the rod and frame test. Piagetian theory predicts that the ability to master the principle that the water level

remains horizontal is gained by about twelve years of age, but it appears that this is much more likely for boys rather than girls (Harris L.J., 1978). Harris shows that there is a substantial difference in favour of males at the sixth grade of high school and at college level in the U.S.A. in this particular facility, even when the response is simply to select a drawing from four options. Harris suggests that boys are ahead of girls in this test from an early age but only significantly so by the fourth grade of secondary education in the U.S.A. This difference highlights the weakness of the Piagetian single unified set of principles approach to developmental psychology which was discussed in the previous chapter but supports the findings of other tests on spatial ability.

The results of these tests, together with tactual discrimination, map reading and left/right discrimination tests, lead Harris (1978) to conclude that males have decidedly better spatial skill than females and "...on a number of tests, only 20% to 25% of females exceed the average performance of males" (p405). However, it does appear that tests involving the visualisation process in working memory show some small differences in performance before puberty, which increase during adolescence and go on into adulthood, whereas perceptual tests show no difference in results with younger children, the differences only occurring at around puberty. In his review of research on spatial ability in men and women,



Nyborg (1983), asserts that Witkin's research using the rod and frame test and Nyborg's own research show a decline in spatial ability in postpubertal girls. He goes on to review the considerable evidence to support the view that spatial performance in postpubertal girls varies with menstrual cycle, being highest in the low estrogen phase, a point which will be returned to in the following section.

With this apparent superiority in spatial skills in males, which is particularly evident in adolescence and adulthood, it is perhaps unsurprising that a difference in performance on mathematical questions which appear to depend upon spatial skills has been detected by Wood and later by Bradberry. However, issues which arise from the evidence discussed above makes the implications for mathematics teaching far from clear. There is the issues of whether differences in performance on these tests can be attributed to conditioning or whether there is evidence to support the idea that differences are dependent upon physiological attributes of the male and female brain.

Linked to this there is the problem of whether differing forms of experience in schools can affect the ways in which boys and girls develop the skills discussed. Importantly, there is the issue of whether the superior skills in symbolic/visual memory tasks and ability to attend to different stimuli more successfully in girls can be utilised in an effective way to enhance the



acquisition of concepts. These issues will be discussed in the following sections.

#### 5.4 Sex differences in cerebral function and brain asymmetry.

It has been established that the left cerebral hemisphere normally controls speech and other verbal processes, whereas spatial ability is critically dependent upon the right hemisphere. It could then be postulated that the findings of psychometric tests which indicate a better performance in visuo-spatial tasks in males and a better performances in verbal skills in females indicates differences in the relative sizes or efficiency of the two hemispheres in males and females. However, the effects of cerebral lesions confined to local areas one hemisphere only do not support this assumption.

Butler (1984) describes how after left hemisphere lesions men are more impaired than women on tests of verbal function, but after right hemisphere damage men are more impaired in visuo-spatial tasks than women. These findings support the argument, not that males and females have differing strengths in right and left hemisphere functions respectively, but that males rely on the left hemisphere for certain verbal functions and on the right hemisphere for spatial skills to a greater extent than do females. This is consistent with the views of Levy (1974)

who has postulated that males have strictly lateralised verbal and spatial functions, while in females verbal functions have encroached upon the capacity of the right hemisphere to the disadvantage of spatial abilities.

This notion of greater brain asymmetry amongst men than women has been investigated by research with men and women who had suffered localised brain lesions, by testing after sodium amytal had been injected into the internal carotid artery (which inactivates one hemisphere), by split field tachistoscropy, and by examining the alpha rhythms of each hemisphere during specific activities, (the alpha rhythm is suppressed during cerebral activity).

Each of these methods of investigation has proved problematic, not least in finding activities which can be assumed to be entirely reliant on spatial awareness or entirely dependent on verbal skills. Once more Gazzaniga's (1985) contention that trying to isolate mental systems in the brain does not illuminate the nature of cognition appears to be borne out. Another major problems has arisen in finding a wide sample of subjects for investigation.

The results of such tests have not been universally accepted as reliable but Jeannette McGlone (1980) in a comprehensive review of the "impressive accumulation" of available evidence from medical and psychometric research concludes that the data "... do not overwhelmingly confirm that male brains show greater functional asymmetry than



females. However, when sex differences are found, the vast majority are compatible with this hypothesis" (p226)

McGlone goes on to confirm that more is known about sex differences in language representation than those in spatial representation, with verbal asymmetries, suggesting left hemisphere dominance, appearing more marked in male than in female right-handers. However, she affirms that the available data suggest the possibility of greater right hemisphere dominance for non-verbal material in males than in females. McGlone is keen to point out that the brain structure for males and females is not dramatically different, stating, "...one must not overlook perhaps the most obvious conclusion, which is that basic patterns of male and female asymmetry seem to be more similar than different". (p226)

It has been suggested that pre-natal gonadal hormones may have a marked effect upon what may be regarded as typical male or female behaviour. Melissa Hines (1982) reports that females with Turner's Syndrome, and therefore deficient in gonadal hormones, show "extremely feminine patterns of cerebral organisation" whereas women who had been exposed to hormones which had been administered to their mothers to prevent miscarriage were more strongly lateralised for verbal stimuli than were their unexposed sisters. Hines concludes that these results are consistent with a "masculinization or defeminizing role for pre-natal hormones" (p71).



This view is supported by Flor-Henry of Alberta Hospital (1980) in a response to McGlone's review. He states that the effects of the more lateralised male brain are independent of cultural influences but are age dependent, becoming more evident at puberty. The origin therefore lies, according to Flor-Henry, in the androgen generated neohormonal interactions which determine both anatomical and cognitive sexual differentiation. He refers to this process as "an embryological priming that becomes reactivated at puberty". (p235)

Nyborg's review shows findings consistent with Flor-Henry's assertion, showing that men with protein deficiency are feminised by surplus estrogen and score like normal women on spatial tasks, and that pre-natal exposure to androgen like agents is associated with enhanced postpubertal spatial ability in women, whereas exposure to surplus estrogen is associated with depressed spatial ability in men.

Stuart Butler (1984) makes the important point that though the evidence best fits a model of greater asymmetry of brain function in males, it is difficult to distinguish between the degree of lateralisation and the degree of readiness to deploy the major cognitive strategies of one or other hemisphere. Research by Earle and Pickus (1982) reinforces this opinion by reporting that males and females had different alpha rhythms when given batches of mental arithmetic problems which differed in difficulty.

Males in the sample developed alpha asymmetry only during the difficult tasks whereas with women the asymmetry was present during the easier tasks but not the more difficult tasks. This evidence clearly support the notion that men and women may select different cognitive strategies for similar tasks.

This "readiness to deploy" or "readiness to resort" to particular strategies reinforces the role of the executive control aspect of information processing with regard to the function of working memory.

Diane McGuinness (1976) proposes that initial differences in sensory perception, with infant girls more responsive to auditory stimuli involving a wide range of tones and pitches, and infant boys more responsive than girls to visual intensity leads to the later differences in verbal and spatial skills. This does not conflict with the brain asymmetry theory, for McGuinness does not suggest that the sensory differences are due to conditioning, being evident at a very early age. However, McGuinness believes that by emphasising cognitive skills at appropriate times then the initial inclination to neglect certain aspects of cognition can, to some degree, be rectified. It is this belief which lies behind her view that the initial superiority in verbal skills shown by girls is not evident in later childhood because schools insist on verbal skills as a means of communication, and boys have no option but to develop this aspect of

cognition. She believes that a similar emphasis on visuo-spatial tasks would allow girls to develop the visuo-spatial aspect of cognition.

There is evidence (Harris L. J., 1978) that training can affect performance in visuo-spatial skills in children. Harris quotes evidence of children given a special programme of instruction, involving techniques such as paper folding and manipulation of solid objects over a period of three weeks. The experimental group improved significantly on a space relation test, compared to a control group who had attended the normal geometry periods, and the girls in the experimental group at the least held their own with the boys in the experimental group. Other evidence from the training of adults is quoted by Harris to support this notion.

### **5.5 Summary and implications for the research.**

There is no doubt that the evidence from the results of G.C.E/C.S.E/G.C.S.E examinations and the A.P.U. secondary surveys indicate a superiority in mathematical performance in favour of boys, and that this superiority is most evident in the higher ranges of ability. Further, this superiority is greater at the age of sixteen than at eleven particularly in aspects of mathematics judged to be dependent upon visuo-spatial skills or involving ratio.



There is a considerable accumulation of evidence to suggest that males are superior to females in visuo-spatial tasks and that the superiority is particularly evident in adolescents and adults, whereas females demonstrate a superiority in verbal fluency and develop reading skills earlier.

The weight of evidence available on brain asymmetry would suggest that the male brain is more asymmetrical than the female and there is also evidence to suggest that the difference in brain asymmetry may be due to pre-natal gonadal hormone activity. This asymmetry can be assumed to lead to more specific reliance on left hemisphere activity for males in verbal, symbolic activity, whereas the female may have access to right hemisphere neural activity for some language as well as visuo-spatial tasks. The superiority of females in tasks requiring a verbal response to stimuli given in terms of pictures of objects or words would support this idea.

Differences in spatial ability appear to be exaggerated in adolescents because, as Nyborg suggests:

most boys reach an asymptote in spatial ability around puberty while a few show a decline. At puberty, some girls reach an asymptote while the majority decline to a pre-pubertal level. (Nyborg H., 1985, p102)

This exaggeration, and the variation in performance in postpubertal girls at differing stages of the menstrual cycle, further supports the view that spatial ability is

affected by levels of estrogen, supporting Flor-Henry's statement that initial embryonic differences are given a 'priming' at puberty.

It appears likely that the selection of cognitive strategies may exaggerate initial differences in neural networks. This is the point made by Butler when he speaks of the difficulties in distinguishing between the organisation of cerebral functions and the readiness to deploy certain strategies. This view, that initial differences may result in the executive control selecting strategies which have been formed in early childhood but which may respond to teaching, fits in well with the evidence that both boys and girls respond to training in visuo-spatial tasks. Wood's evidence from examination questions that some schools were able to eliminate differences between boys and girls on questions for which boys were much more successful over the the whole sample also supports this view.

It is suggested in this thesis that the teaching of trigonometry often leads to the identification of visually cued procedures without necessarily promulgating the relational understanding of the enlargement and rotation necessary to understand the conceptual basis of trigonometry. Given the better performance of girls in language based work during the primary school years it is possible that girls select a procedural strategy, associated with written structure, when the mathematics



they meet has a relatively low level of demand for spatial links. This would concur with the A.P.U. findings on the areas of primary school mathematics in which girls were superior. If this strategy is then associated with success it is likely that, even more than boys, they will be searching for such procedures as a means of dealing with trigonometry questions.

In this respect, and given the relatively poorer performance in visuo-spatial tasks associated with girls at this age (14-15 yrs), introductory lessons using poor techniques to explain visuo-spatial relationships are likely to result in a poorer awareness of these relationships amongst girls than boys. As detailed in the last chapter reliance on procedures which are not meaningful can lead to great problems in dealing with questions which are variations on those previously encountered and can result in poor performance as decay and interference leads to forgetting.

It could be argued that when these factors are taken into account, the greater emphasis on spatial relationships in the mathematics undertaken by high ability children may explain the increased superiority of boys in the high performing bands of ability. It would certainly be expected that a reliance on instrumentally learned procedures would lead to a feeling of being unable to understand 'what the subject was about', with an



accompanying insecurity and rejection of further study when it became optional.

The computer based approach is based on the integration of information given in spatial and numerical forms, in which girls may have an advantage over boys. The ability to integrate information given in different forms may suggest that girls already have an advantage in becoming more versatile learners, as has the greater ability in females to attend to more than one stimulus. The encouragement to perceive visual patterns, linked to numerical forms should increase the capacity to use visual imagery, which in general is poorer in post-pubertal girls than in boys.

By this means it may be possible to encourage visuo-spatial skills in girls and the readiness to deploy visualisation, building trigonometrical ratios on a sound conceptual basis. There is also the confidence aspect to be taken into consideration, for as stated above, girls may underrate their ability to succeed in mathematics. If girls do reach an understanding of the visuo-spatial basis of the ratios then one would expect that confidence levels would increase leading, to greater motivation for the development of the topic.

The research will measure the performance of boys and girls in the experimental groups as separate samples with the null hypotheses:

1. That there will be no difference in performance between the girls in the experimental groups and the girls in the control groups, and,

2. There will be no difference in performance between the boys in the experimental groups and those in the control groups.

The hypotheses will be tested using two tailed tests of significance as the alternative hypothesis will not predict a superiority from either sample. (This of course requires larger differences in the mean scores to achieve 5% significance levels than if a one tailed test were used)

The research design and details of the tests to be used will be discussed in the following chapter.

## 6. Research Design

### 6.1 Preamble

The research structure is that of a classical comparison between control groups and experimental groups, using a statistical technique, one-way analysis of variance, to test the hypotheses at a 0.05 level of significance. In addition, the establishment of matched pairs of control and experimental groups allows the non-parametric test, the Wilcoxon matched pairs test, to be applied for boys and girls together and for boys and girls separately. Following this, a selection of individuals from both experimental and control groups were interviewed, the results recorded and transferred to a written transcript in an attempt to illuminate the results indicated by the written tests.

Given the major thesis that a computer graphics approach would aid the conceptual understanding of trigonometry, there was an obvious need to assess the extent to which a conceptual understanding had already been gained as a result of previous teaching. With this in mind a pre-test was given to all groups at the outset of the research. However, two factors were to be considered when assessing the relative merits of the experimental approach: the first was that of skill in the more routine 'procedural' mathematics associated with this topic and considered important by both teachers and students, the second was that of longer term retention of both conceptual



understanding and procedural techniques. It was thus decided to give two post-tests approximately two months apart and to sectionalise these tests into questions designed to assess a more conceptual over-view of the topic and questions based on the more routine procedures which were heavily emphasised by the teachers of the control groups.

In addition the second theory, that girls in the experimental groups would gain more from the experimental treatment than girls in the control group would gain from the control treatment, would be tested by analysing the results, using one-way analysis of variance, for both of the post-tests, and the non-parametric Wilcoxon matched pairs test. The null hypotheses would be: that there is no difference in performance between the girls in the experimental group and the girls in the control group, and, there is no difference in performance between the boys in the experimental group and the boys in the control group. It is recognised that an alternative hypothesis could have been that the girls in the experimental groups would perform better than the girls in the control groups, thus instigating a one-tailed test and increasing the likelihood of a statistically significant result being achieved, but at the outset of the research the possibility of girls rejecting an approach based on linking visualisation to numerical coding had to be taken into account and it therefore appeared more valid to assume a non-directional alternative hypothesis.

The particular format of the written tests was designed to allow the relative performance of the two groups to be compared for each individual question, if required, as well as for each of the two sections and the overall total. In addition the relative performance of girls and boys in both experimental and control groups could be analysed separately for each of the two sections of the two post- tests. The interviews, which would take place approximately two months after the second post-test, were designed to assess possible differences of approach, and the ability to utilise visualisation, between experimental and control subjects, as they attempted to solve trigonometric problems. The students would be asked to verbalise their thinking as they attempted to solve more basic questions and when presented with apparent inconsistencies between information given in visual and numerical form.

However, this section of the research can not be viewed as a rigorous statistical exercise as the students were given different levels of encouragement and help in order to facilitate the discussion, and no quantitative assessments were made. Rather, it is an attempt to discover whether the assumptions made about the way in which the students answered the questions on the written tests were borne out by the students descriptions of what they were thinking as they attempted trigonometry questions some time after they had been taught the topic.

## 6.2 The school



The research was carried out in Kenilworth School, a 12-18 comprehensive school in a prosperous dormitory town for Coventry and Birmingham. The school was formed in 1976, by the merging of the previously existing two secondary modern schools and the grammar school into a three hall federal school, consisting of two parallel twelve to sixteen halls, on adjacent sites, and a sixth-form hall on a site some mile and a half away from the main campus.

The federal system is rather unusual in the English secondary education system, existing at the time of the research in central Warwickshire, Banbury in Oxfordshire and Milton Keynes in Buckinghamshire, and indeed it appears to be fast disappearing. Banbury School has reverted to a more orthodox system and Kenilworth itself is in the process of changing into a single unit 12-18 school. The initial intention of federal schools was to allow children to be part of a relatively small institution, the hall, but to share the benefits, in terms of teachers and resources, of a much larger institution, the school. Thus the teachers would be free to teach in any of the halls, and each teacher would belong to an academic faculty which had responsibility for the curriculum throughout the school, though he or she would have a pastoral responsibility within a hall which had its own head and pastoral team. When entering the school the pupils were encouraged to think of themselves as joining a particular hall which had its own identity.



In this part of Warwickshire the age of transfer to the secondary system is twelve plus, rather than the more usual eleven plus, so that pupils have only four years in the secondary system before sitting the recognised examinations at sixteen plus. However, in order to make comparisons between this school and others with the more usual age of transfer less open to confusion, pupils in the initial year in the school are termed 'the second year'. This research was carried out with pupils at the beginning of their 'fourth-year', who were one year away from the year in which the external examinations would be sat, so all of the pupils were aged 14-15, with the great majority aged 14.

The school is situated in an area where housing is more expensive than in much of the surrounding district and is generally regarded as a desirable place to live. The verbal reasoning quotient scores on entry show a mean of 111, (the mean score for the population as a whole being 100), with approximately 15% of the intake attaining scores of 120 or higher.

There is a high parental expectation of success in national examinations and the school achieves levels of success well above the national average. In mathematics, the school regularly achieved a pattern of some 40% of its fifth year attaining a grade A,B,C, at G.C.E. ordinary level or C.S.E. grade 1, compared with the 25% of the population which the Cockcroft report gives as the national figure, and this figure has increased to 44% with the G.C.S.E. examination (grade C or

above). Approximately 12% of the fifth year would be expected to go on to follow the G.C.E. 'A' level courses in mathematics, with an average pass rate of 85%. However, the school has very few girls following the 'A' level course in Pure and Applied Mathematics, rarely more than 25% of the two groups, though girls would typically make up around 50% of the Pure Maths. with Statistics course.

On entry to the school every effort was made to balance the intake, in terms of ability, between the two halls. However, there is some evidence that parental choice has led to the hall which was the former grammar school receiving a slightly higher number of able pupils. The external examination results, over a number of years, showed a slight superiority for this hall, though not significantly so. This hall was used for the control groups for the research

The mathematics department consisted of nine graduates in mathematics, or a mathematically related discipline, and one teacher who had retrained to teach mathematics. In addition the two heads of halls, both qualified as science teachers, taught some mathematics.

### 6.3. The teaching groups.

On entry to the school the pupils were assigned to a tutor group in one of the two halls. Care was taken to establish mixed ability tutor groups in both halls and these became the basis of the teaching groups for the academic year. The school



at the time of the research had been eight form entry for some years, having fallen from a twelve form entry at its formation. Four tutor groups were assigned to each hall. After one term the pupils were assigned to ability groups for language teaching but remained in mixed ability groups for the all other subjects. No special group existed for 'remedial' or 'slow-learning' children but a Special Needs department existed to give support to the teacher in the classroom. The same teaching structure was followed into the third year, (their second year in the school), but at the end of the third year the pupils were placed in ability groups for mathematics in the fourth and fifth years.

The teaching of the mixed ability groups in the second and third years was based on individualised work schemes which the teachers built up over a number of years. It operated by dividing the syllabus into a number of 'modules', which were studied for a period of three weeks before moving on to the next module. Each pupil was given a module sheet consisting of book page references, question numbers and work sheet references in the order in which they should be attempted. The teachers introduced topics with a lead lesson but the pupils were then able to work at their own pace through the material. Each pupil was issued with two text books in the third year (Oxford Comprehensive Mathematics Series, Green Book 3 and Blue Book 3), from which large section of the work was taken but this was supplemented by worksheets, which were professionally printed and issued to all pupils.



The module sheet contains some references which were coded to show that they should be taught to the class as a whole. These were typically a range of practical activities which were more efficiently organised on a whole class basis). Other references referred to material suitable for the very able pupils or as reinforcement material for pupils needing more consolidation of work.

As each class followed the same module sheets for two years it was relatively easy for the staff to formulate tests to assess progress through the course. These tests together with end of year examinations were used to build a profile of the pupils' performance over the two year period. The tests and examinations were written at three levels of difficulty to allow pupils to succeed, so some initial estimate of pupils performance was necessary before the examinations were sat. At the end of the third year, on the basis of test scores and teacher opinion, the pupils were divided into ability groups to prepare for the external examinations. Each hall then had five ability sets of comparable ability, though the very small fifth set in each hall contained pupils of low ability for whom a special course was developed. The first four of these ability groups from one hall, the former grammar school, were used as the control groups, with the equivalent groups in the second hall used as the experimental groups.

The structure of the teaching groups in this school thus offered several distinct advantages for this research. The first major advantage was the knowledge that previous

experience in the second and third years was consistent for the control and experimental groups. The second advantage was the detailed knowledge of the teaching of trigonometry which had been consistent for all third year groups. The third distinct advantage was the detailed knowledge of the testing which had been the basis of the formation of the experimental and control groups, giving a picture of the relative performances of the corresponding groups over a number of topics before the pre-test was given.

The experimental groups will be coded E1, E2 ,E3 and E4 and the control groups will be coded C1,C2, C3, C4, with the numerical values signifying the 'set' (1, being the highest ability group, 4 the lowest), throughout the remainder of the thesis. The teachers of the control groups will be coded Mr.A (C1), Mr.B (C2), Mr.C (C3) and Mr.D. (C4), with the researcher, who taught all of the experimental groups coded Mr.E.

#### 6.4 The teachers.

The teachers of the control groups had all taught mixed ability third year groups in the previous year so they were aware of the previous experience of the pupils. As they taught in both halls some of the teachers had been assigned to more than one fourth year group. Mr A. was also the normal class teacher for E3 and Mr. B. was the class teacher for E4, whereas, Mr.C and Mr.D. only taught one class each.



Mr. A. was a teacher of some twelve years experience, seven of these in the present school, who had been given the responsibility for administering the faculty policy in that particular hall. As such he was given the title of 'head of department'. He is a popular teacher with both staff and pupils, and has had experience of teaching mathematics throughout the 12-18 age range and across the spectrum of ability. He is an honours graduate in mechanical engineering but has taught mathematics exclusively throughout his teaching career. He is known for his relaxed manner with pupils and favours individual or group work from prepared material rather than employing a black-board based didactic approach to each lesson.

Mr.B. is a very experienced mathematics graduate who had been teaching in the hall when it was a grammar school. He has taught throughout the age and ability range for a number of years and had, until the year of the research, been responsible for administering the external examination system. Mr.B is noted for his concern for pupils and his approachability.

Mr.C. was in third year of teaching at the time of this research, having joined the school after completing his P.G.C.E. as a secondary mathematics specialist. He is an enthusiastic teacher who is very keen on group work and discussion based on prepared work-sheets, though he often introduces his lessons with a short didactic input using the black-board.



Mr.D., the teacher of C4, was in his fourth year of teaching, at the time of the research having spent two years teaching at another Warwickshire comprehensive school. He is recognised as a very good mathematician who also has a great interest in information technology. He is a keen computer programmer and manages the school networks. His background as a pupil in a successful but unconventionally informal comprehensive school in Warwickshire, in which he also completed his teaching practice when undergoing his P.G.C.E., has left him with a strong inclination towards problem solving and group work activities for which he carefully prepares appropriate materials. His relationship with the teaching groups is relaxed and his quiet yet energetic manner appears to be appreciated by all pupils, but in particular those pupils who are not considered mathematically able. Mr.D. was aware that trigonometry was not part of the recommended syllabus for pupils intending to take the lower level papers in the G.C.S.E. examination but he was more than willing to try to teach this topic by introducing a carefully prepared set of activities aiming to achieve a conceptual understanding. As will be seen in the next chapter, Mr. D's approach to the topic bears a good deal of resemblance to the approach taken by the researcher, though of course he did not have access to the dual coding computer graphics.

All of the experimental groups were taught by the researcher who is a mathematics graduate with higher degrees in education. He taught mathematics for fifteen years before being

employed as lecturer in mathematics education at Birmingham Polytechnic. For seven years he had been employed at Kenilworth school as head of department in the sixth-form centre and head of mathematics faculty, so two of the teachers, Mr.A. and Mr. B. were known to him before the research began. Although obviously familiar with the structure of the school and the mathematics courses, none of the children taught by the researcher were aware that he had previously taught at the school as he had left before they entered as second years.

The arrangements for teaching the groups were made with the principal of the school and the current head of the mathematics faculty,

#### **6.5. Previous experience of trigonometry**

Each of the mixed ability third year groups had experienced part of a module on trigonometry, which was to be introduced by a class lesson before exercises from work sheets and from Oxford Comprehensive Mathematics were attempted. This module would occur toward the end of the summer term of the third year, just before the end of the year examinations, approximately twelve weeks before the teaching for the research began. In practice, it would be known that some members of each third year group would be destined for the lower sets in the fourth year and the teachers would direct these pupils towards finishing other modules rather than attempting the trigonometry questions. It would be expected that around 20 of the 28-30



pupils in each group would have started the trigonometry module with approximately 5 from each group finishing the whole set of exercises. In this way the material was intended to extend the most able pupils, some of whom would be preparing to take the external examinations a year earlier than normal.

The introductory lessons, based on the Oxfordshire Comprehensive Mathematics books use the unit triangle, with the rotating hypotenuse to define the sine and cosine of the angle, which is the basis of the computer based approach. The students would thus have been introduced to the sine and cosine and answered problems based on the scaling of the unit triangle. The text book approach has been carefully designed to build up the concepts, using well drawn accurate diagrams, identifying the sine and cosine of angles by the colour coding of the opposite and adjacent sides. All pupils were encouraged to use a calculator rather than books of tables. but three figure tables were available, and the text book itself supplied three figure values of the sine and cosine for angles from 0 to 90 degrees in 10 degree intervals which were suitable for most of the questions encountered. The books also supplied several well thought out problems requiring the techniques established earlier, with the problems being accompanied by diagrams.

In addition to the text book, three specially prepared work sheets were given to pupils who had successfully attempted the text book work. These sheets provided a reminder of the definitions of the sine and cosine of an angle, using the triangle with unit hypotenuse. It also gave the method of



finding the length of the opposite and adjacent sides in terms of ' $r \sin A$ ' and ' $r \cos A$ ', with  $r$  being the hypotenuse. The exercises on this sheet went on to ask the pupils to find missing sides and angles in unit hypotenuse right-angled triangles. The second sheet asked the pupils to find missing sides or angles in right angled triangles where the hypotenuse was not of unit length and the third sheet used embedded triangles or isosceles triangles and expected the pupil to find or construct the relevant right angled triangles, as well as introducing seven word problems which were not accompanied by diagrams illustrating the problems.

Each third year teacher would assume that pupils who were likely to enter set 1, in either hall, would have completed most of the text book work and some or all of the work sheets. The pupils who would attempt the hardest level third year paper would have attempted the book work, though not necessarily the work sheets, so the majority of each of the second sets in the fourth year had been introduced to the sine and cosine of angles and attempted questions based on the techniques described. The majority of E3 and C3 would have been introduced to the topic from the text books but no trigonometry question appeared on the intermediate tests taken by pupils who were likely to enter set three or set four. The pupils in E4 and C4 would have been unlikely to have been introduced to this module in the third year. A 'lowest level' test was given to those pupils who may be entered for the lowest level of G.C.S.E. papers, but who had been in the past considered as non-

examination pupils. Two pupils in C4 and one in E4 took this test.

#### 6.6. The Pre-test

The pre-test was intended to allow pupils throughout the ability range (from sets 1 to sets 3) to answer some questions based on their previous experience and included two questions which required a basic understanding of similarity which for some pupils could be more a matter of intuition than a recollection of previous knowledge. A further question, the third, asked for a recognition of the definition of the sine of an angle used in both the text book and the work sheet (a similar question was later to be used in post-test two), and the fourth question required an understanding, which again may be intuitive, of the effect on the sides of the unit right angled triangle of enlarging the angle. In this case the triangles were not drawn accurately so the student was expected to note the angle sizes from the diagram and use this knowledge to assess the correct option for the length of one of the other sides.

The remaining questions were all based on the kind of questions the students would have encountered in their previous work, though two questions, seven and nine, used orientations which were designed to assess how much the students were dependent on a limited prototype of the right

angled triangle, and question ten was designed to test only the most able students.

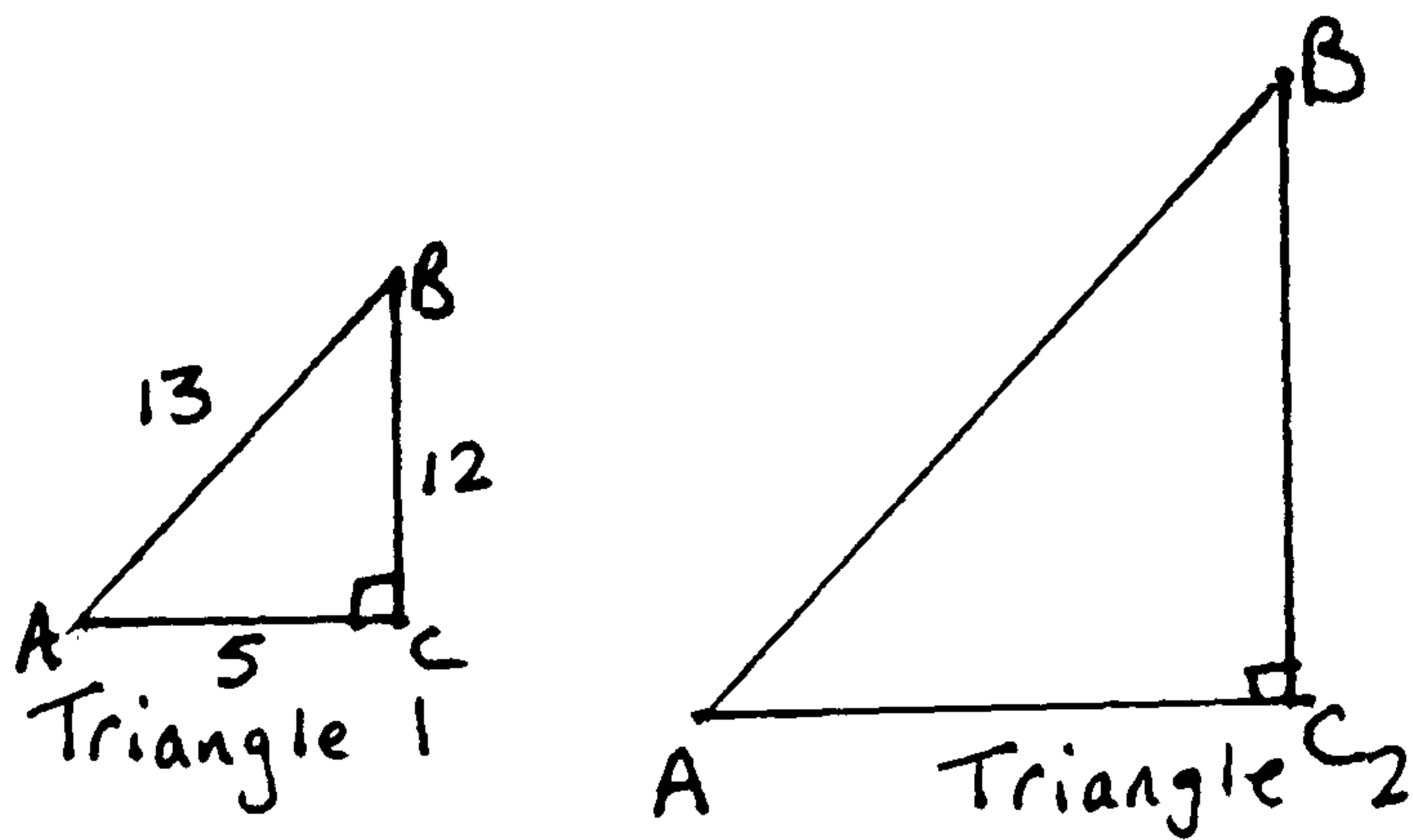
The class teachers of E3 and C3, thought it proper to give a week of revision to their students, using the third year material, as their students would not have achieved as much in this module as those who entered the higher sets. The teachers of E1 and C1 were more confident that their pupils would be able to answer some of the questions, though they expected to teach trigonometry again at this point in the fourth year. Mr.A. was very impressed with the pupils of C1 and felt that though they may need some further teaching the majority of the class could master trigonometry. The teachers of E2 and C2 expected the pupils to answer only the first few questions but were interested to see how much of the third year work had been understood. They did not revise the topic before the test was given. It was agreed that there would be very little point in giving the pre-test to E4 and C4, but the teaching would begin assuming that there had been no previous experience.

The pre-test is given in full on the following page.

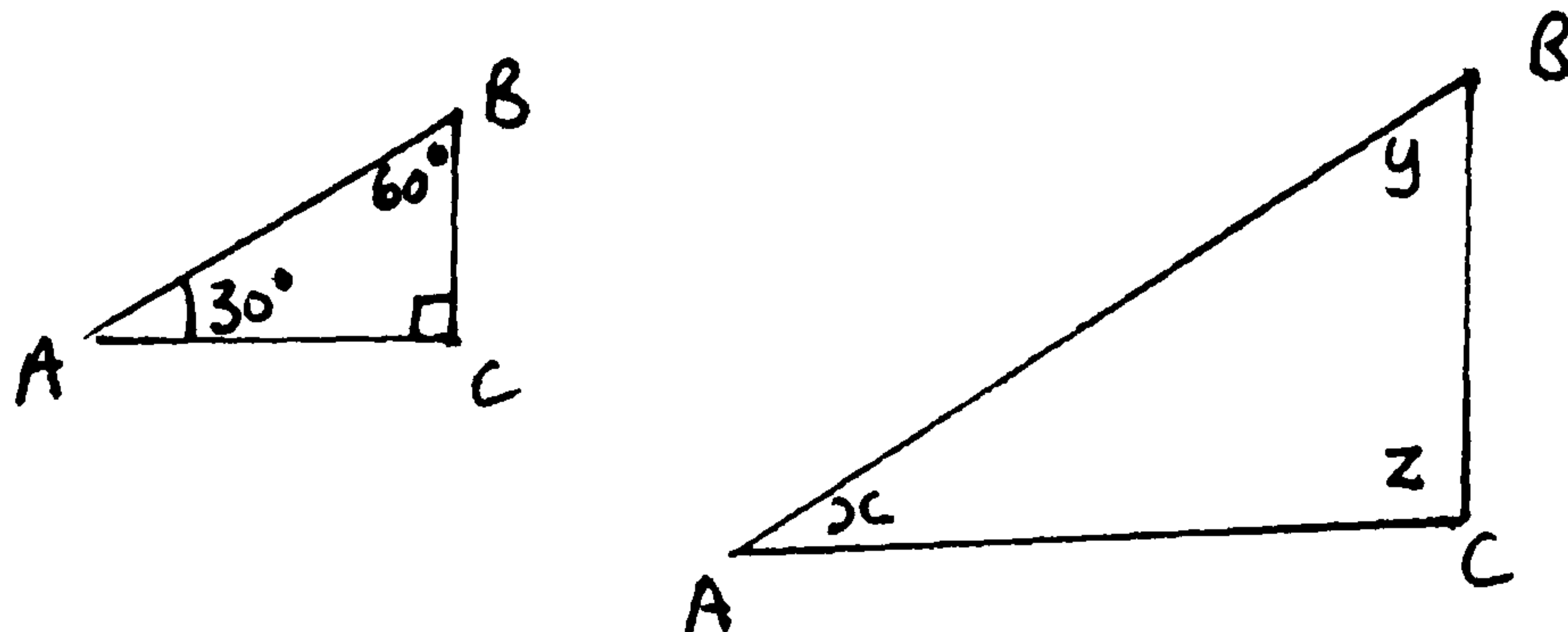


# Pre-Test

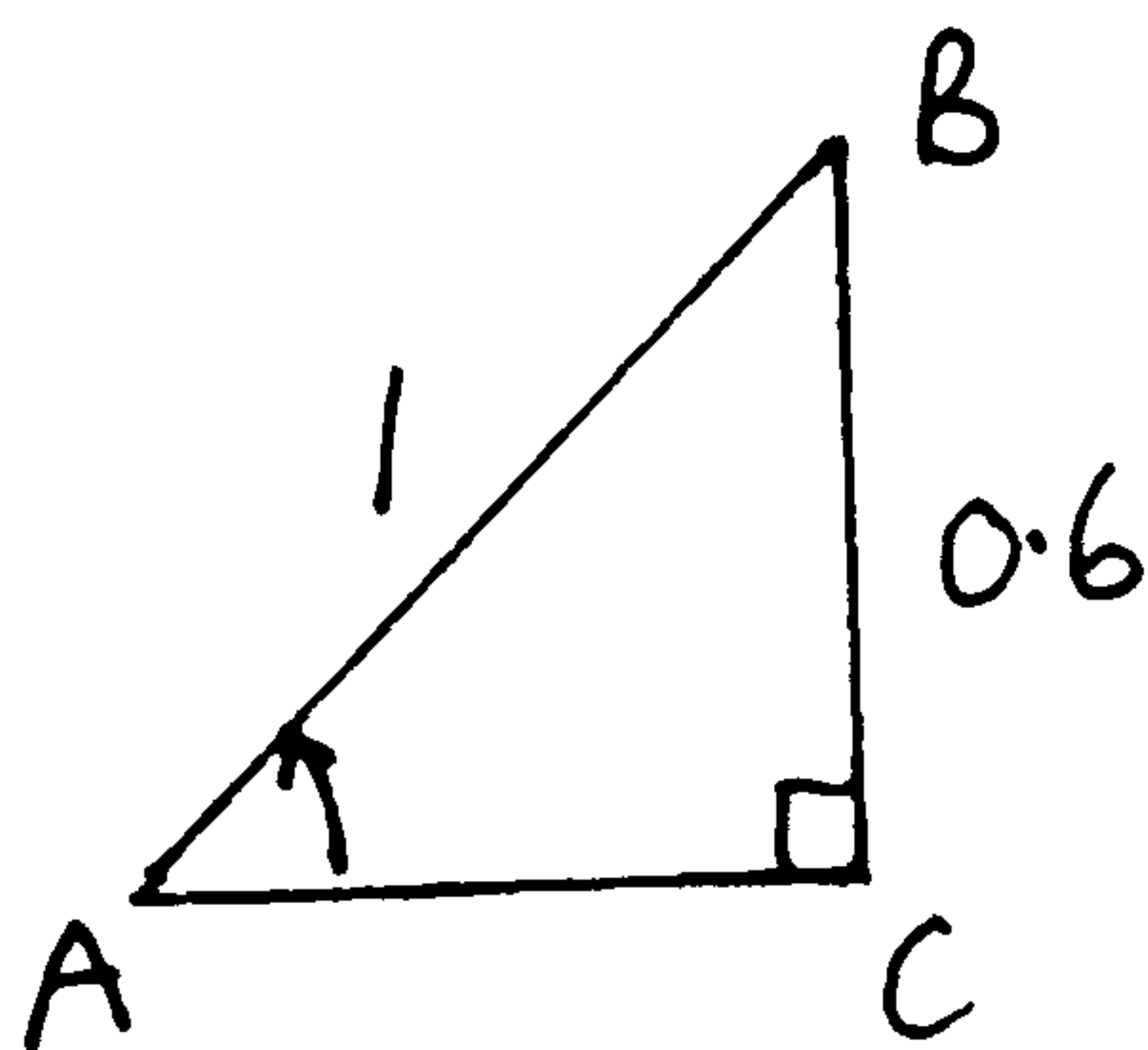
1. Triangle 1 has been enlarged by a factor of +2 to give triangle 2. What are the lengths of the sides of triangle 2?  
(Write  $AB=$  ,  $AC=$  ,  $BC=$  )



2. Triangle 1 has been enlarged by a factor of +2 to give triangle 2. What are the missing angles in triangle 2?



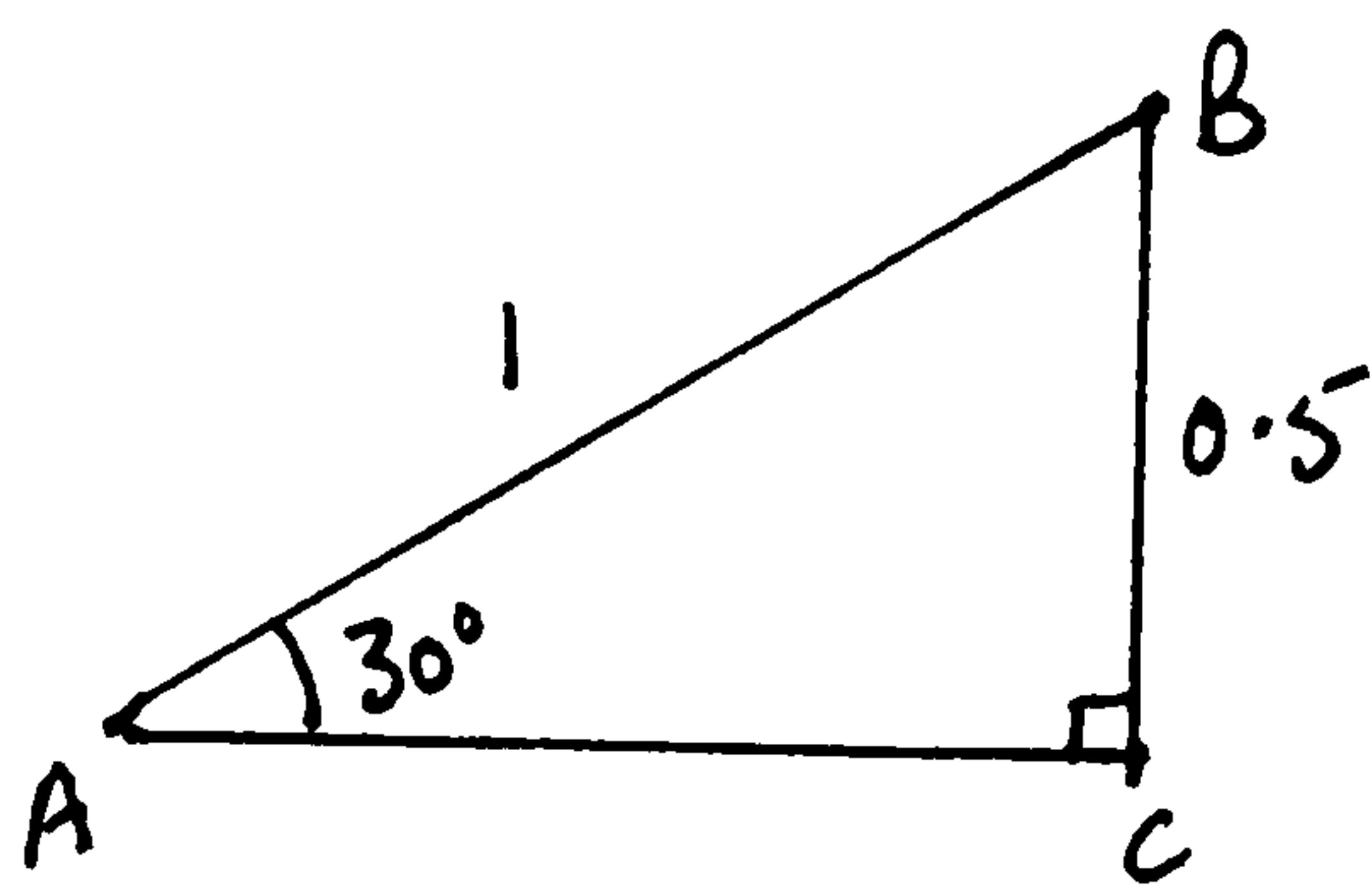
3.



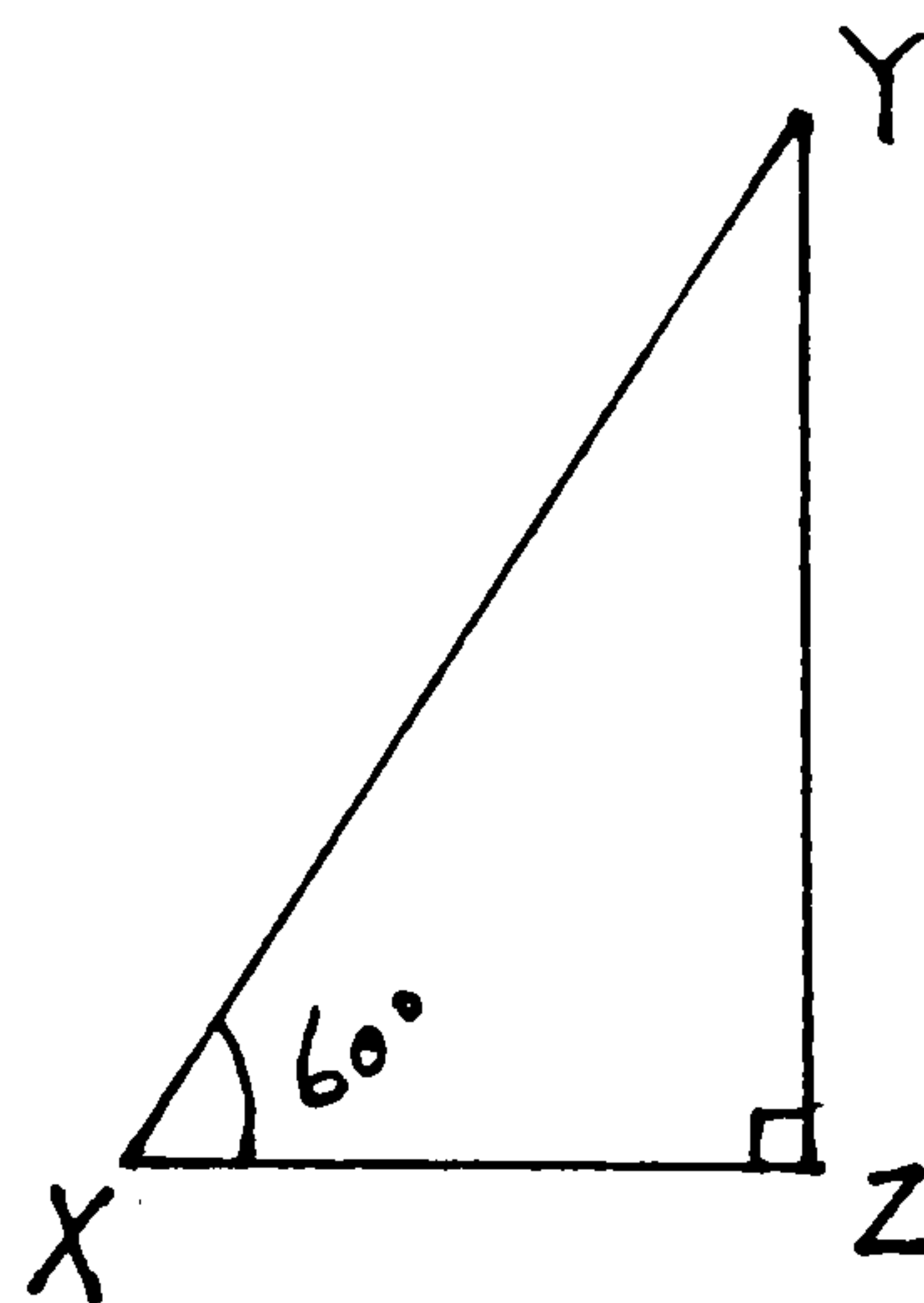
The sine of angle A is:

- a) 1
- b) 0.6
- c) 90 degrees
- d)  $1/0.6$
- e) 1.2
- f) don't know

4. Triangle 1



Triangle 2

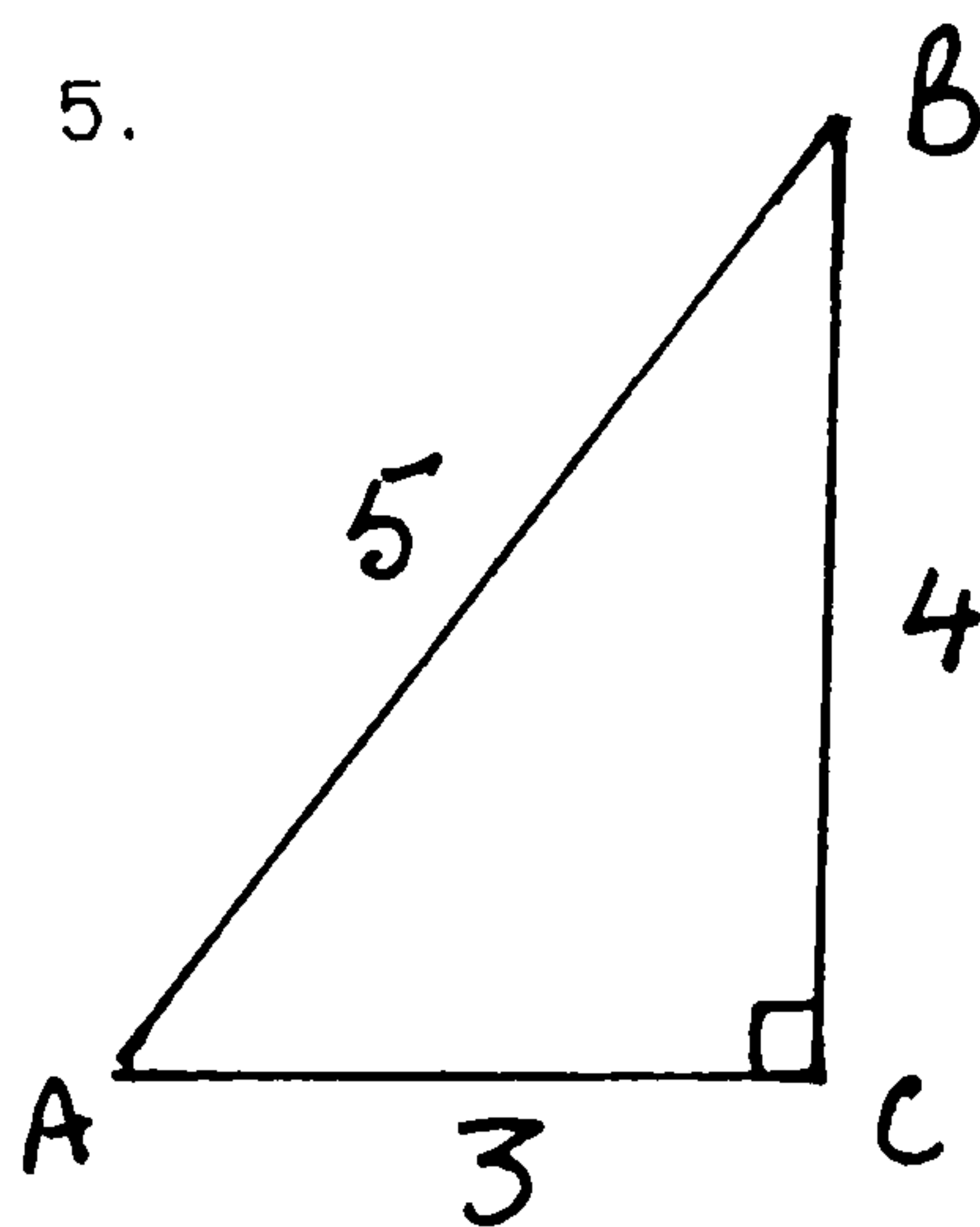


YZ is:

- a) less than 0.5
- b) greater than 0.5  
but less than 1
- c) exactly 1
- d) the same as XZ

Can you write down a side which is the same length as AC?

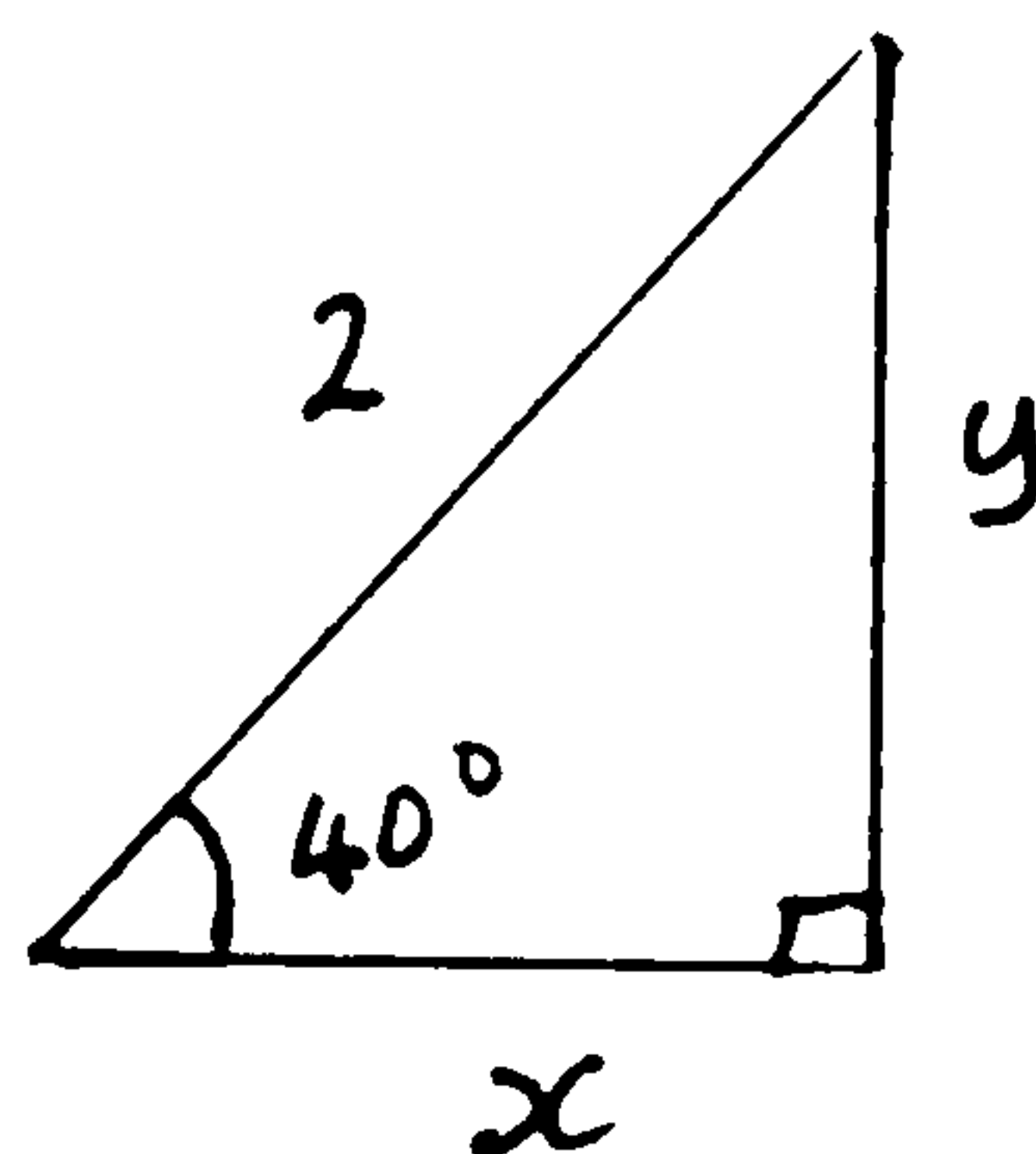
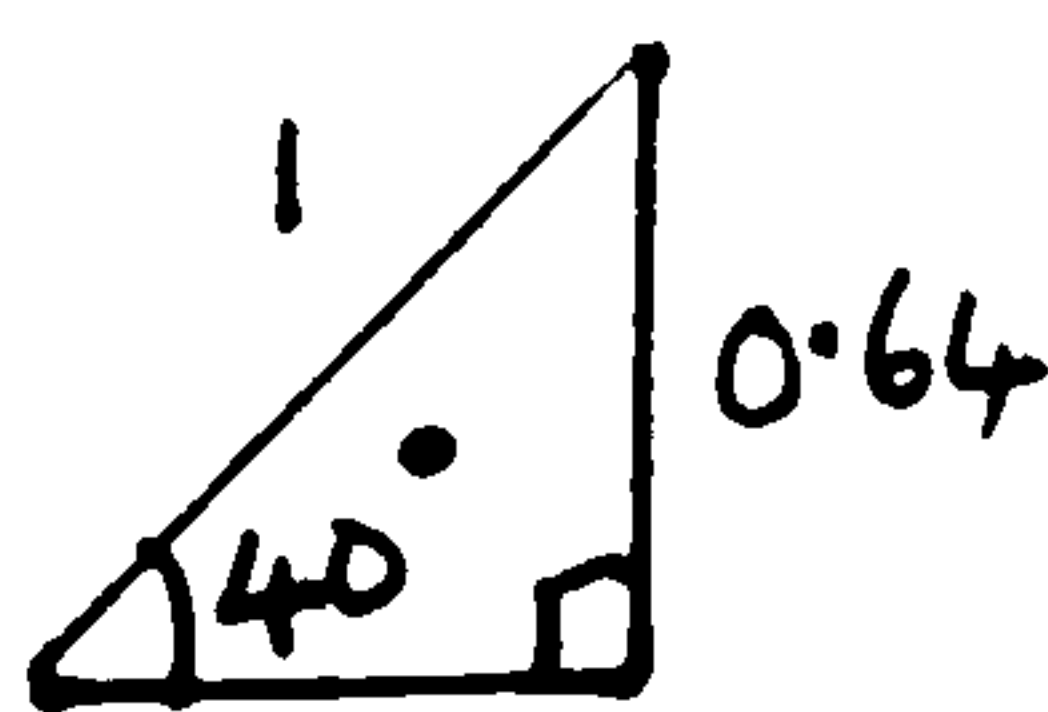
5.



What is:

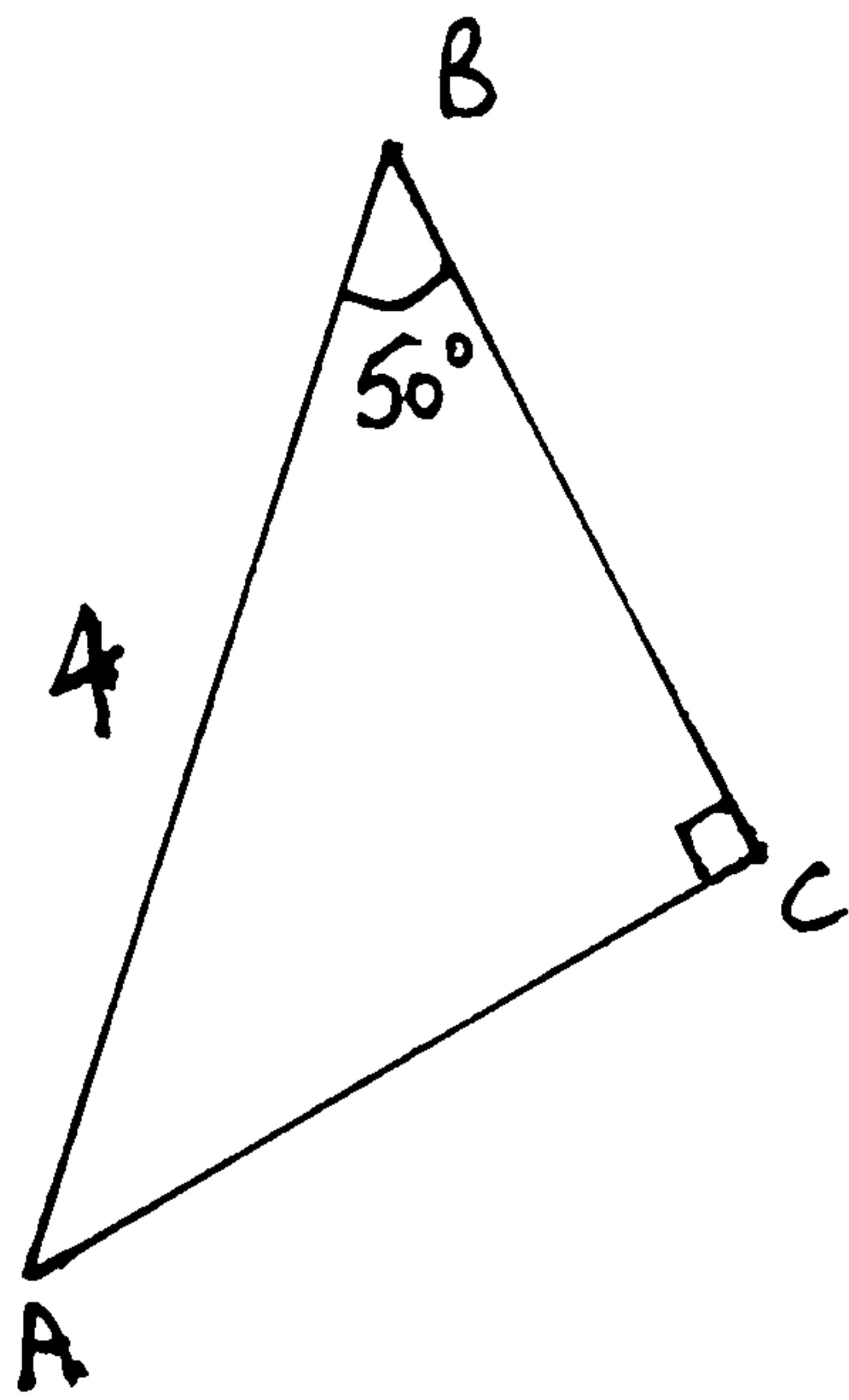
- a) the sine of angle A
- b) the cosine of angle A
- c) the sine of angle B
- d) the cosine of angle B

6.



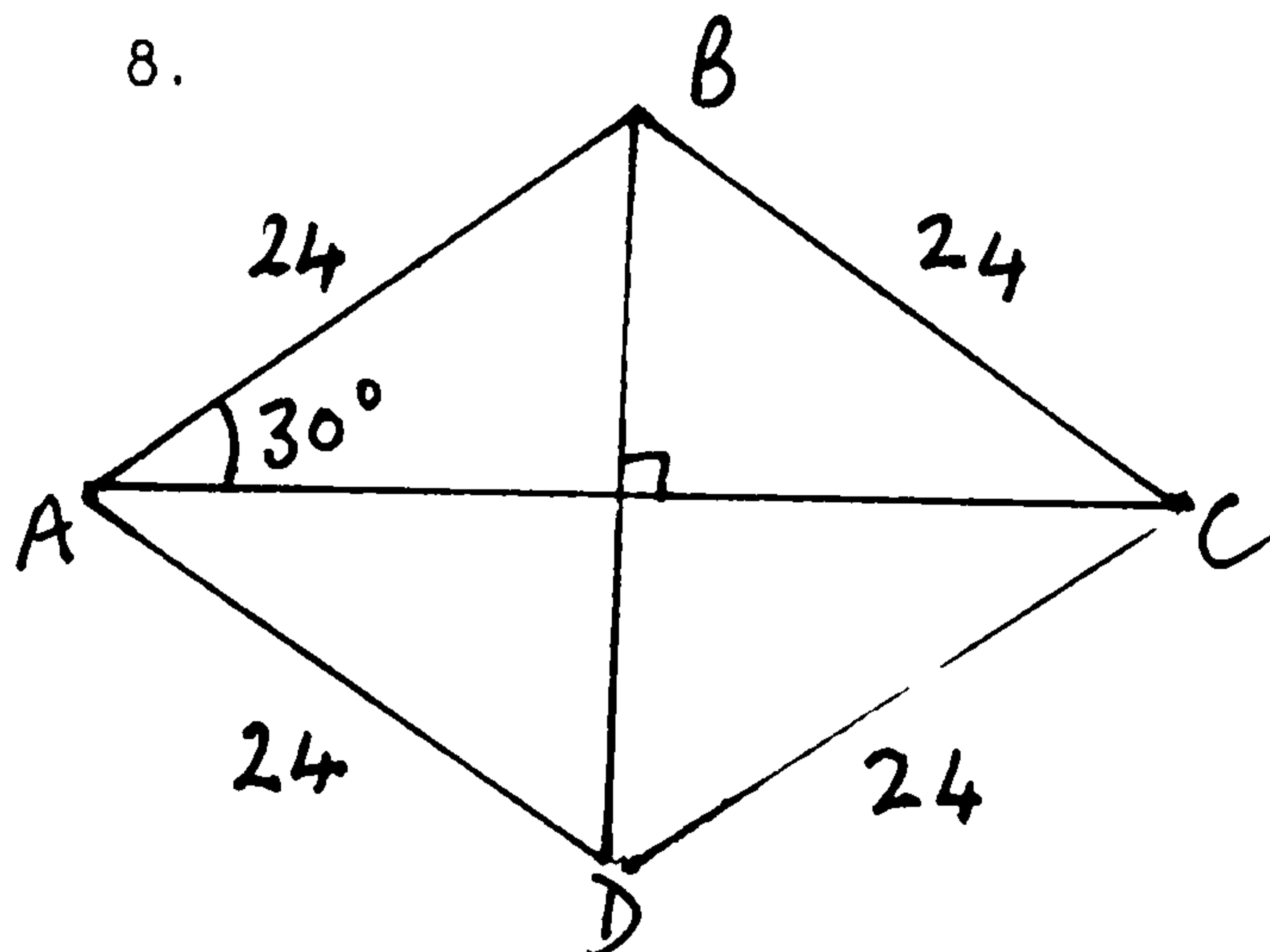
- a) What is the length y ?
- b) show how you can find the  
length x then calculate  
this length (to 3 d.p.)

7.



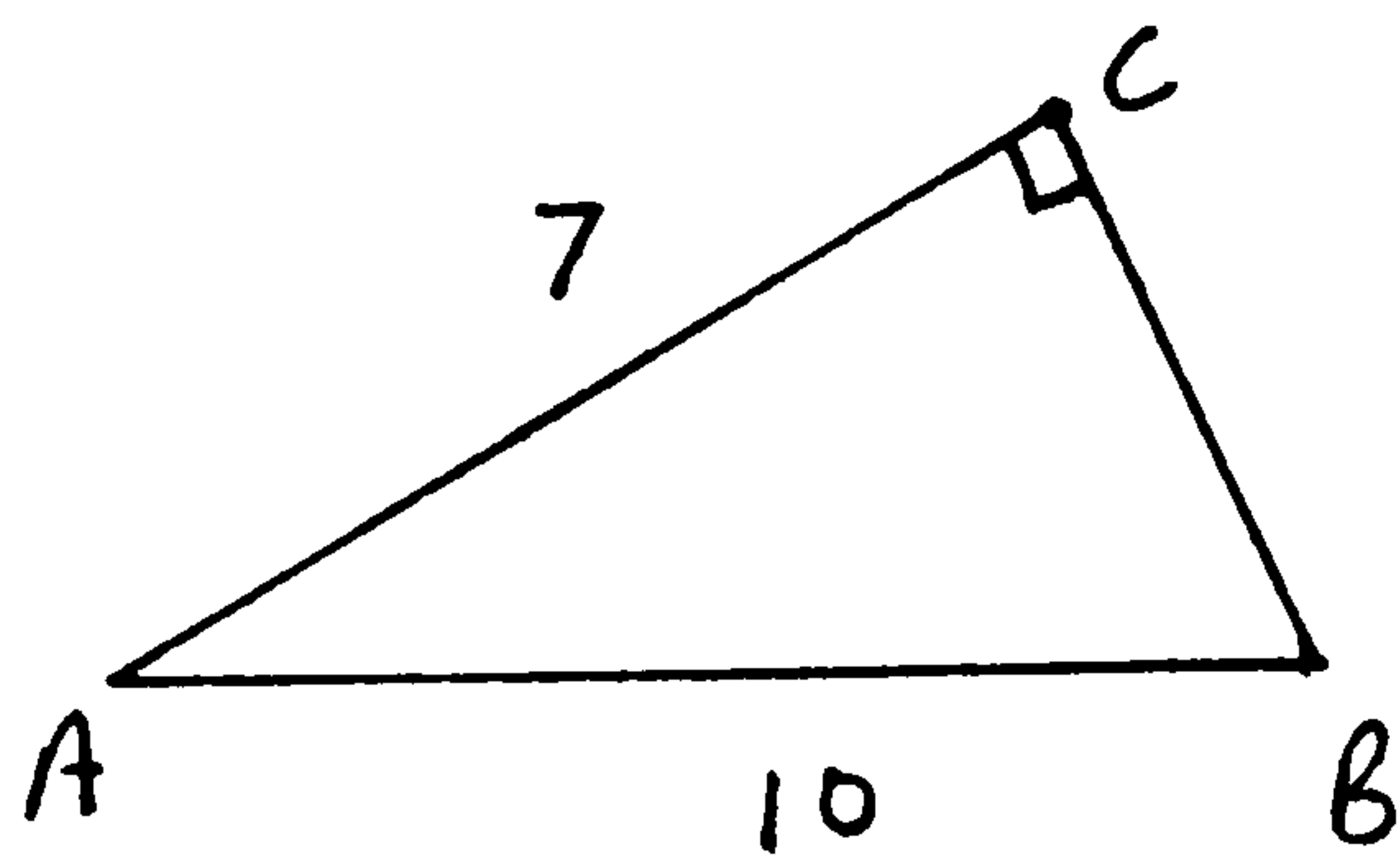
Show how to calculate the length BC then calculate this length ( to 3 d.p.)

8.



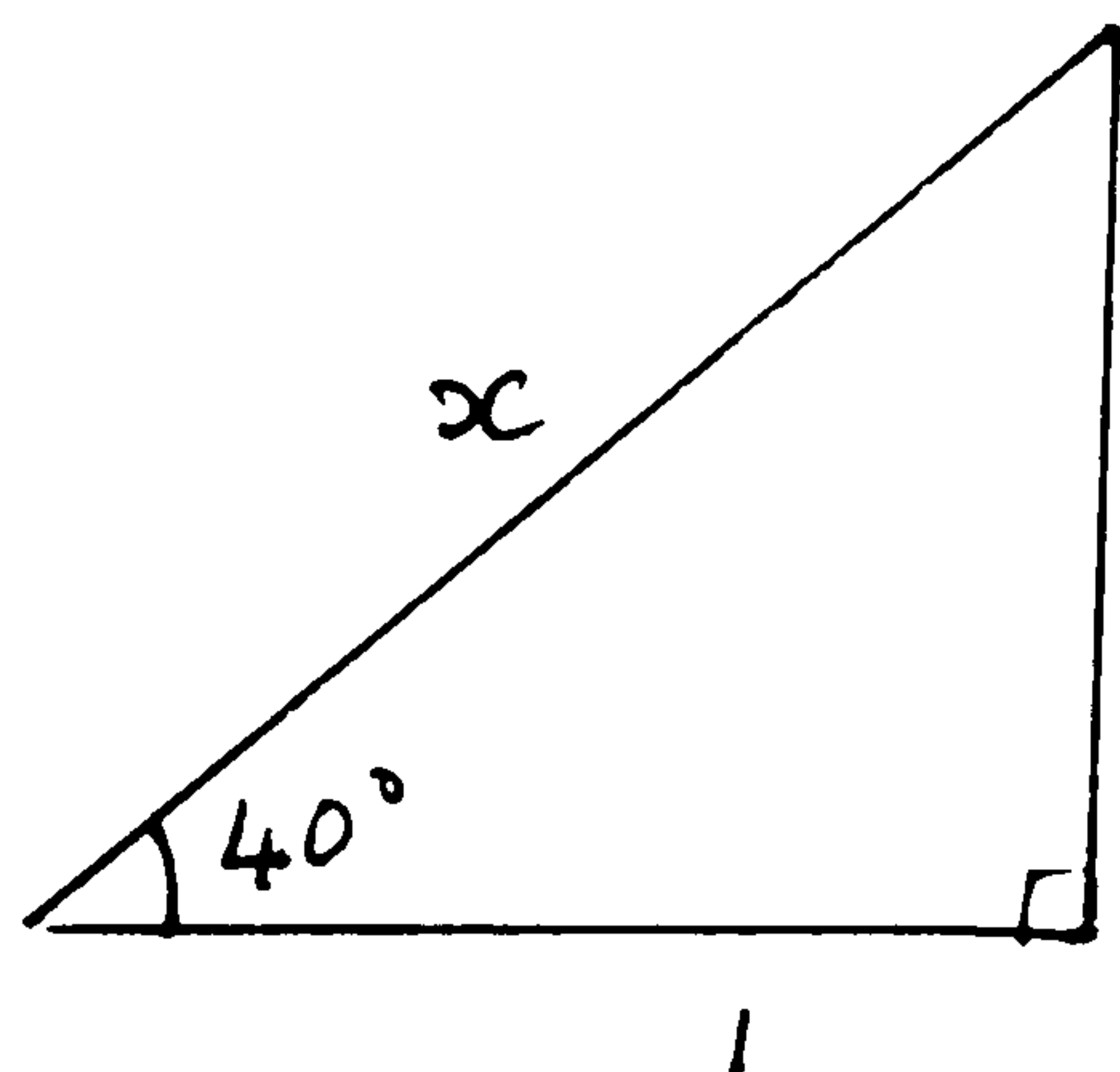
AB, BC, CD, DE are all 24cm. Show how to calculate AC, then find this length.

9.



Show how to find angle A then find this angle.

10.



Show how to find the length x then complete the calculation to find the length.



### 6.7. Results of pre-test

The students were asked to comment on the test when they had finished noting for example whether they found it easy or difficult and if they found it difficult why this should be the case. They were informed that all comment would be treated as confidential by the researcher and the fullness and frankness of the comments made by students from all the groups which took the test were very revealing. Some comments will be quoted but there were far too many to quote all of these in full. There were feelings which were common to a great many of the pupils in each set and these will be discussed at the end of every pair of corresponding teaching groups.

The marks for each question will be given together with the percentage of each class achieving correct answers. As it was possible to achieve marks for parts of some of the questions the totals and averages will not correspond a simple multiplication of number gaining correct answer by the number of marks for the question. Marks displayed for each question answered correctly are intended to give an idea of the ways in which the sets differed in their capacity to answer the range of questions given. The numbers in brackets refer to the marks awarded for a correct answer to the question

E1 (N=25, 9 boys, 16 girls)

Q. 1(1) 2(1) 3(1) 4(2) 5(2) 6a(1) b(2) 7(2) 8(3) 9(2) 10(3)

% 100 96 60 31 12 81 32 44 16 0 4

Mean mark: 36.7, boys' mean 44.4, girls' mean 33.1

C1 (N=29, 13 boys, 16 girls)

Q. 1(1) 2(1) 3(1) 4(2) 5(2) 6a(1) b(2) 7(2) 8(3) 9(2) 10(3)

% 97 93 50 37 33 80 30 30 23 0 8

Mean mark: 38.5, boys' mean 44.2, girls' mean 33.8

These two sets representing the highest ability groups, all of whom had studied the topic ostensibly to a higher standard than the questions in this test demand. However, it can be seen that there are distinct problems in recognising the definition of the sine of the angle (Q3), the very basis of the approach used in their previous teaching and problems associated with the calculations involved in Q6b, 7 and 8. In particular the different orientation of the triangle used in Q9 meant that no member of either group could identify the ratio to be used. Most simply wrote that they did not understand this question. Only one member of E1 and two members of E2 doubled the angles when enlarging triangle one to triangle two (Q2), though several members of both sets produced nonsense answers to Q3, including a number who selected 1 as the correct answer. Again, in Q4, where the student was required to use the information from the first diagram to make inferences about the second.

there were several selections of 'exactly 1' as the correct value for the opposite side, which would make it the same length as the hypotenuse. Several students exhibited pro-active inhibition, with the right angled triangle cue prompting a response, not from the more recently learned trigonometry frame, but from the earlier Pythagoras' theorem frame, despite the fact that the latter could not be applied to the questions in point.

The written messages from students in these two sets were very similar and centred on what they perceived to be their inability to remember this topic. Some were more self condemnatory than others, with such statements as, " I'm sorry, I know I have been taught this but I have completely forgotten it all" (Jennifer, C1 ). Others wished to make the point that they had needed extra help with this work when they first encountered trigonometry with such statement as, "my sister taught me how to do these but I can't remember it all" (Ross, E1). Members of both groups, five from E1 and 6 from C1 made specific references to the difficulty of the topic and pointed out that they had not understood it, with comments such as, " I was taught this last year but I never really understood it" (Richard, E1). All of the comments and marks from C1 were relayed to Mr. A., the class teacher.



E2 (N=25, 12 boys, 13 girls)

Q.	1(1)	2(1)	3(1)	4(2)	5(2)	6a(1)	b(2)	7(2)	8(3)	9(2)	10(3)
%	100	88	16	16	0	80	0	0	0	0	0

Mean mark: 17.9, boys' mean 17.9, girl's mean 17.9

C2 (N=26, 15 boys, 11 girls)

Q.	1(1)	2(1)	3(1)	4(2)	5(2)	6a(1)	b(2)	7(2)	8(3)	9(2)	10(3)
%	100	92	46	12	12	62	4	4	4	0	0

Mean mark: 20.1, boys' mean 22.2, girls' mean 17.3

It can be seen that Q1 and Q2, involving the more intuitive enlargement concepts, were answered well, though three members of E2 and two members of C2 doubled two of the angles when the enlargement of factor 2 was made in Q2. Approximately 50% of each group selected the option 'exactly 1' for the opposite side in Q4, presumably assuming that doubling the angle has the effect of doubling the side opposite to it, without apparently recognising any cognitive conflict with the fact that the hypotenuse, of one unit length must be larger than the other sides. The majority of each group was able to achieve a mark for Q6a, recognising that when the hypotenuse is scaled by a factor of 2, the opposite side is scaled by the same factor, but only one member of C2 and none from E2 was able to use this scaling process, employing the cosine of the angle to calculate the third side of the triangle. The use of trigonometry to find missing sides or angles was clearly not understood by members of either group.

The comments made were again similar to those made by members of E1 and C1, with most students recognising that they had completed some work on trigonometry but that they had forgotten it. However, there were clearly some doubts in the mind of several students with phrases such as, " I think I might have covered this but if so I have forgotten it all" (Richard, E2) and more students accepting that they might not have understood trigonometry in a relational sense though they had worked out answers to questions, illustrated by Samantha's comment, "We were taught this but I can't remember how to work out the answers now, I never understood it very well"

Mr.B., the teacher of C2, received the answer papers and comments, recognising that there was very little which could be assumed to be understood about trigonometry.

E3 (N=21, 8 boys, 13 girls)

Q.	1(1)	2(1)	3(1)	4(2)	5(2)	6a(1)	b(2)	7(2)	8(3)	9(2)	10(3)
%	100	67	48	19	5	86	0	0	0	0	0

Mean mark: 20.3, boys' mean 21.6, girls' mean 19.6

C3 (N=21, 11 boys, 10 girls)

Q.	1(1)	2(1)	3(1)	4(2)	5(2)	6a(1)	b(2)	7(2)	8(3)	9(2)	10(3)
%	100	67	52	19	5	71	5	10	0	0	0

Mean mark: 21.1, boys' mean 24.1, girls' mean 17.8

The week of 'revision', which for some of the class would have been an introduction to trigonometry allowed these sets to have

mean marks similar to E2 and C2. The general distribution of marks is much the same for these sets, except for the fact that there were a third of each of the two sets who doubled some or all of the angles for the enlargement of the triangle in Q2.

The written responses contained some differences from the other groups, with several members of each group clearly disturbed at their own more intuitive responses to Q1 and Q2. Approximately 25% of each of these groups made statements about these questions along the lines of, " I couldn't remember this so I just guessed". Many pupils said that it was a difficult test reporting that they had forgotten it all, (this despite the fact that they had been taught only a few days previously), but there were some heartfelt comments about the nature of the topic such as, " I don't understand it because it is too hard and we never got taught how to do these" (Sharon, C3)

Mr.C. received these comments and marks with the acceptance that he would need to re-examine the introduction to the concepts.

As previously stated, no pre-test was given to either E4 or C4.

There was evidence from the responses from all sets that pupils had already associated this topic with a set of procedures to learned and that they had simply forgotten the procedures. In some cases this would result in pro-active inhibition, with an attempt to apply Pythagoras' theorem by squaring angles! Answers which it may have been assumed cause cognitive conflict were frequently selected, (Q2, Q4),



indicating that the cognitive frame was limited and did not incorporate the wider geometric implications of the result being selected. In these cases, a numerical answer was not related to a visual image; the diagram was as Fischbein suggested (see chapter 4) a symbolic device rather than a primary intuitive instance.

#### **6.8. The research structure in detail**

The term 'control' is used by statisticians in at least two different ways. A 'control' subject is one who did not receive the treatment and an experiment is 'controlled' when the researcher determines which subject will receive the treatment. In this research both uses of the term are valid. There are teaching groups who do not have access to the computer based approach to link numerical and visual representation and are therefore 'control' subjects and the research is more than an observational study for the researcher can select which groups are to receive the experimental treatment. This selection is not on a random basis but on the basis of knowledge of previous achievement and assumptions of comparative ability based on testing which took place at the end of the third year. In this sense the structure of the school makes the allocation of control subjects relatively easy. The results of the pre-test support the end of year test in the similarity of performance in mathematics between the control groups and the experimental

groups. The following table gives end of year test for each group together with pre-test scores:

End of Year Score		Pre-test (All)	(Boys)	(Girls)
E1	61.3	36.7	44.4	33.1
C1	69.9	38.5	44.2	33.8
E2	47.1	17.9	17.9	17.9
C2	48.4	20.1	22.2	17.3
E3	46.5	20.3	21.6	19.6
C3	48.8	21.1	24.1	17.8
E4	39.1	Not Given		
C4	38.9	Not Given		

Note that E3,C3, E4, C4, were given different end of year papers than E1,C1, E2,C2, so the closeness of the results between E2, C2 and E3, C3, do not signify a similarity of ability. However, the teachers in the school believe that there is a great difference in ability between the majority of pupils in E1, C1 and E2, C2 which is borne out by the end of year tests and the pre-test results.

Research such as this in the educational sphere is naturally beset by problems of confounding effects on the assumed principal factor assumed to be the cause of variation

in results. In this case the confounding effects have been reduced by the following considerations:

1) The social background and the previous experience of the control and experimental groups is bound to be very similar as all subjects are from the same locality which itself is relatively homogeneous.

2) The teachers of the control groups are well established, experienced teachers who know each other and the children in both the experimental and control groups and the timing of the research, two weeks after the start of the new summer term, would mean that differences in expectation would have had little time to take effect following the mixed ability third year grouping.

It can be always be claimed that the 'Hawthorne Effect', where being the focus of new working methods and research attention may generate improvement in motivation and output, is likely to be a factor in educational research where experimental groups know they are receiving some form of special attention. In this research the Hawthorne effect is minimised by the fact that pupils in both experimental and control groups are equally aware that the researcher will mark and note progress on the tests given after the teaching has been completed, but neither the experimental groups, nor the control groups would be aware that the experimental groups would be receiving different teaching from the control groups. Though the experimental groups would be aware that they are being taught



by a different teacher, they would not be aware that the control groups were not also being taught by the researcher, nor would they realise that their performance was being compared to that of other groups. In the same way, the members of the control groups would not be aware that their performance was being compared with others, but that the test results were being analysed to assess how much progress they were making in trigonometry

It should also be noted that the experimental groups would experience some disadvantages by the organisation of the research. The use of the computers demanded that pupils in E2, E3 and E4 had to change from their normal teaching room to the one in which the three computers could be temporarily be set up for teaching. These computers could not be left permanently in situation because of security problems and because the room was constantly in use with other teaching groups. The positioning and setting up of the three B.B.C. Masters and disc drives at the beginning of many of the lessons would often mean some disruption and loss of time from these lessons. Not least of the difficulties was that the researcher was not known to the pupils as a teacher but as someone who arrived, often with little time to spare, taught the lessons and left again. This meant that there was little time to establish a sound teacher pupils relationship or even to know the pupils' names. It can be seen that this situation would not help with routine problems of classroom management and control.

The classical research methodology would favour a double blind experiment, where neither subjects nor evaluators know which subjects are in the control or experimental groups. Though, as described above, some efforts were made to reduce the pupils awareness of whether they were in control or experimental groups, there could be no way of attempting to keep the evaluators ignorant of which pupils were in which groups. However, safeguards built into the research were that papers marked by the researcher were given back to class teachers who in turn returned the papers to the pupils. This had the effect of having the marking checked by experienced markers and even more rigorously by the pupils themselves.

The classical research structure would also suggest that the order of events would be:

Pre-test → Treatment → Post-test → Delayed Post-test  
with no further treatment between the post-test and delayed post-test. However, in educational terms, where the success of the teaching is not limited in its effects to research findings, but to the future success in mathematics of the pupils concerned, it would be difficult to justify the giving back of papers which indicate problems and expect the teacher to ignore these problems for the sake of the research, especially when the pupils are likely to ask about incorrect answers. For this reason, the control groups would have a small input of teaching between post-test 1 and the delayed post-test 2. The experimental groups would have only the return of the papers and a general comment on the results as the classes



would again be taught by their usual teachers and the researcher would only return to give a five-minute summary as the papers were examined. More detail about the input made between the two post-tests will be given in the chapters devoted to these two tests and the results. However, it should be noted that any advantage from additional teaching would tend to accrue to the control rather than the experimental groups.

The results will be analysed using analysis of variance for a two sample case. This technique consists of obtaining two independent estimates of variance, one based upon the variability between the two groups and the other based upon the variability within groups. The ratio between these two variance estimates can be measured for significance using Fisher's F distribution. The variance estimates are found by finding the within group sum of squares and the between group sum of squares and dividing each by the appropriate number of degrees of freedom. For the between group variance estimate the number of degrees of freedom is one less than the number of groups, which as the number of groups will be two will give the degrees of freedom as 1, and the within group degrees of freedom will be the number of subjects in both groups minus 2. The between group variance divided by the within group variance produces the statistic F which can be tested for significance. For all aspects of this research the level of significance will be taken as  $p < 0.05$ . For the two sample case the analysis of variance is equivalent to the Student t test ( $t^2 = F$ ) and the level of significance given is for a two tailed test, that is



no prediction is being made in the alternative hypothesis about which group will perform better. The null hypothesis will be that there is no difference in the performance between the two groups in question, the alternative hypothesis being that one group will perform better than the other. (A significant F ratio leading to the conclusion that the groups were not drawn from a population with the same means)

The analysis of variance as a technique assumes that the variances for each group are homogeneous, that is, drawn from the same population of variances. This assumption can be tested and where the assumption is invalid, (the ratio of the two variances is also an F distribution which can be tested for significant differences) adjustments will be made to the technique. Where the variances of the two groups differ significantly but the numbers in each sample are close, the analysis of variance is a conservative test and will be used in its original form. If the number in each group differs considerably then adjustment suggested by Edwards, (Edwards A.L., 1968, p102) will be applied.

In addition to this parametric test it may be suggested that a test which makes no assumption about background population would lend itself well to this kind of research. A non-parametric test can be applied to research of this kind and the Wilcoxon signed ranks matched pairs test appears particularly appropriate. Sixteen pairs of subjects were matched according to their scores in the third year test and their scores on the pre-test, both having to be close, with any

difference being in favour of the control group, for the pair to be acceptable. The pairs were also to be matched in such a way as to allow eight experimental boys to be compared with eight control boys and eight experimental girls with eight control girls. The combination of these restrictions made it difficult to enlarge the number of pairs beyond sixteen, especially as the groups E4, C4, did not take the pre-test and could therefore not be used for the selection of pairs. The actual scores for each pair are given below. The number in each pair code refers to the set number and B or G refers to boy or girl. These pairs will be used to compare experimental and control group subjects throughout the research.

Pair code	A1B	B1B	C1B	D1G	E1G	F1G	G2B	H2B	I2B	J2G
Pre-test% E	50	25	15	40	60	20	20	15	15	20
C	70	25	15	45	60	25	20	15	20	20

Exam. % E	65	60	63	70	60	45	45	53	54	52
C	70	72	65	81	57	45	50	54	56	53

Pair code	K2G	L2G	M3B	N3B	O3G	P3G
Pre-test% E	10	10	20	25	20	15
C	15	15	20	25	20	15

Exam. % E	44	44	36	43	28	25
C	51	45	38	46	28	30

## 7. The Teaching Plan, Teaching and Results.

### 7.1 Introduction

The two-week, twelve lesson, teaching programme allocated to this topic was based on the need to allow pupils in the experimental groups to develop a conceptual understanding of trigonometry, that is a relational understanding of the basic concepts and their place in a wider knowledge representation structure. As this understanding develops, the students would also gain an understanding of, and facility with, the techniques and procedures used to answer associated questions.

A key factor would be the perception of visual patterns which underlie the trigonometric ratios, linked to the numerical coding of the lengths and angled being displayed.

In order to achieve this facility for switching between global, visual imagery and serialistic, verbally encoded procedures the computer graphics would be an integral part of the teaching, and the computers themselves a necessary component of the classroom equipment. A fuller description of the software and the way in which it was incorporated into the general teaching strategy will be given later in this chapter.

The results of the pre-test, given in the last chapter, were such that all of the four teachers of the control groups felt that it would be prudent to begin the teaching programme from the position of introducing the subject as if there was no previous knowledge. If students responded quickly to the ideas



introduced because previous knowledge was being cued then this was all to the good, but the evidence of the pre-test suggested that there were very few students in this position.

The teaching plan for the control groups was negotiated with the four teachers, at least in terms of the objectives of knowledge, skills and understanding. The way in which each teacher approached the teaching of his set was likely to vary with each one employing tried and tested methods. The concepts to be introduced were largely the same for all four groups but the more able pupils were expected to be able to solve problems of greater difficulty, both in terms of interpretation of word problems and in terms of algebraic skills requiring a more formal understanding.

## **7.2 Teaching objectives**

The objectives were that students should:

- 1) Develop an understanding of the concepts of the sine, cosine and tangent of angles in a right angled triangle and the range of values of the sine and cosine for angles between 0 and 90 degrees.
- 2) Be able to use the sine or cosine of an angle to find another side when the hypotenuse and an angle other than the right angle were known.
- 3) Similarly, be able to use the sine or cosine of an angle in order to find a missing angle, where one side of a right angled triangle and the hypotenuse were known.

4) Be able to use the tangent of an angle to find one of the shorter sides when an angle other than a right angle and a side other than the hypotenuse is given.

5.) Be able to use the tangent of an angle to find a missing angle when the two shorter sides of a right angled triangle are given.

6) For any of the above situations, be able to recognise which of the trigonometric relations would be appropriate to employ in order to find the required information.

7) Be able to construct diagrams and solve questions based on the objectives above for problems related to practical situations given in word form, though it was agreed that many pupils in the lower sets would find this a difficult objective and that the teachers would be concerned to match the difficulty of the problems to the ability of the children.

In addition, students in sets one and two, with the possibility of some in set three should:

8) Be able to find the hypotenuse when one other side and another angle were known.

It should be noted that all four of the teachers concerned agreed that the first objective was a necessary pre-requisite for the achievement of the other objectives. In terms of relative difficulty, the teachers agreed that objectives 6,7,8 would present more difficulty to students than the previous 5, though the introduction of the tangent may present some initial problems.

Each teacher would design his own specific objectives for each lesson and the teaching methods employed would be left to the individual on the basis of previous experience. It was agreed that the researcher would visit two of the teaching periods taken by the teachers of the control groups to experience the teaching style of each teacher and discuss the strategy by which he intended to reach the objectives above. At any time the teachers were free to comment on progress and difficulties experienced in teaching this topic, and to visit the researcher as he taught the experimental groups..

### 7.3 The software design.

In order to give students the opportunity to develop a versatile learning approach it was crucial that the computer graphics could produce a clear visual image which would allow the user to perceive the changing values of lengths and angles as variables were inputted, and that these values could be displayed numerically.

If the computer graphics were to improve upon those images produced on the printed page, then the advantage of the dynamic image on the screen was to be involved. The speed at which the computer image is produced would allow students to produce a series of images from which they could abstract the concepts to be acquired. Thus the definition of the sine of the angle as the length of the side opposite the angle in a right angled triangle when the hypotenuse was one unit, and the cosine as



the length of the adjacent side in the same triangle could be linked to the graphical representation of the triangle, with varying angle and corresponding varying lengths of the two shorter sides.

The software specification, around which the programs would be written was:

1. It should provide a triangle ABC with a right angle at C, given the initial input of the values of two sides of the triangle, or two sides and an included angle, or two angles and a corresponding side. The initial layout of the screen would allow the user to move a cursor down a column of side and angle names with the option of inputting the value of the side or angle indicated. Upon receiving adequate information the completed triangle would appear on the screen and the user would have the option of revealing the values of the other sides and angles of the triangle

2. The user must be able to select whether the values of sides or angles of the triangle constructed should be revealed, and these numerical values must be available on the screen with the drawing. The user would thus have a visual impression and a numerical value on screen at the same time. The link between the visual encoding and the verbal encoding would be encouraged by allowing the user to move between the two as required.

3. The revealed lengths or angles should be capable of being displayed using a specified number of decimal places, so that if the hypotenuse was 1 unit, the opposite side and the adjacent side to the specified angle could be displayed at up

to four decimal places, whereas if the hypotenuse was a larger figure the sides may be displayed to the nearest whole number or one decimal place.

4. The user should be able to specify an angle to the horizontal at which the base of the triangle should be drawn, so that the user becomes familiar with right angled triangles drawn in different orientations.

5. The triangle constructed from the initial input variables should be capable of being enlarged by a given factor, redrawn, and the new values of the sides and angles displayed. This multiplying factor should also be capable of being applied to angles, so that it can be seen instantly that multiplying an angle by a given factor does not increase the sides by the same factor. The facility to use the same technique for dividing by a factor should be available.

6. In a similar way, it should be possible to add or subtract a constant amount any specified angle or side and the triangle be redrawn and new values displayed.

7. There should be an optional geometric 'stretching' instruction, to observe the values of angles and sides when one side is stretched.

8. If the triangle, constructed to an initial scale becomes too large for the screen, the user should have the option of applying an automatic rescaling of the drawing so that it fits on to the screen.



8. In order to help with the estimation process, the 'unit' length on the hypotenuse, or on either of the other two sides should be an optional display.

9.. The input variables and the revealed display for the numerical values of sides and angles should be different colours, so that they are easily distinguishable.

10. The geometric and numerical displays should be linked continuously and the user can select to display the numerical values after initial estimates have been made, or as a continuous process as the triangle is transformed.

With this software specification it was possible for a programmer to produce the software used for the research. The initial screen layout was as follows:

**↑↓ select  
& type  
Display  
f0-4 d.p.**

**→AB=?**

**BC=**

**CA=**

**∠A=**

**∠B=**

**∠C=90°**

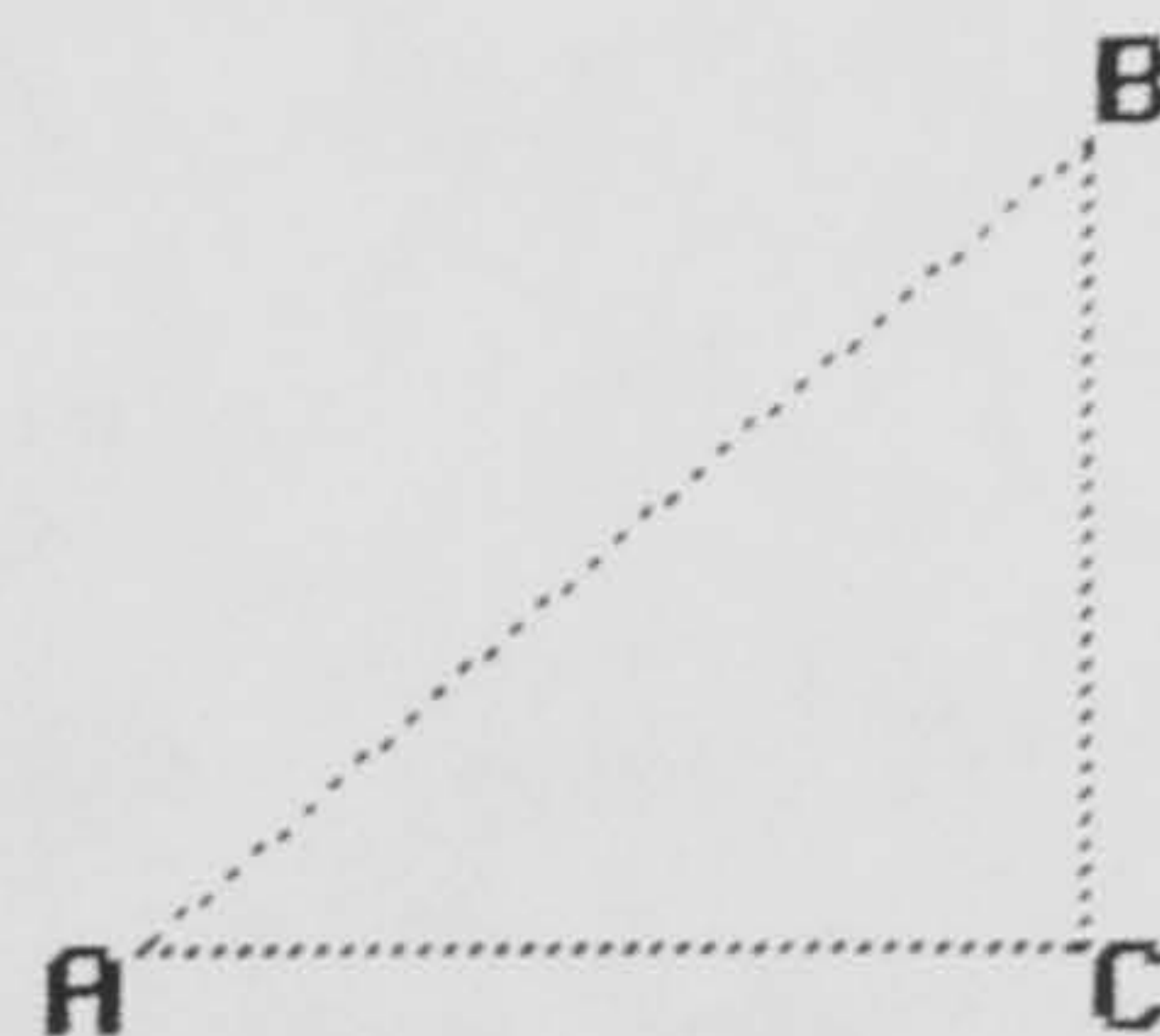
**+ - \* / change**

**Ⓜ 10°**

**Rescale Off**

**↔ stretch**

**Unit End**



The screen is in colour and the initial letters of the instructions on the left hand side are highlighted to signify that touching this letter on the key-board starts the



operation. The cursor arrow on the initial screen cues an input for AB, but it can be moved up or down the screen using the 'up' arrow or 'down arrow' on the key-board. The function keys f0 to f4 select the number of decimal places used for the display, the '@' key, allows the user to input an angle, through which AC on the opening screen will be rotated, and the operation keys +-\* / allow the user to input a number, so that when the return key is pressed the side or angle at which the cursor arrow is pointed will transform. The values of the other sides and angles will change correspondingly and the drawing will change at the same time. The 'Rescale', 'Unit' and 'Stretch' facilities are activated by the initial letter or the arrow keys on the key-board. As soon as there is enough information inputted for the computer to draw the triangle it is constructed on the screen, but the values for the other sides and angles are left blank until the user moves the cursor arrow to a particular side or angle and presses 'D' for display.

The following screen display shows the drawing after an initial value of 5 units has been given to AB, and an angle of 40 degrees given to angle A. The triangle has been rotated through 30 degrees and the unit hypotenuse marked. The cursor has been moved to BC, CA and angle B and these values displayed.

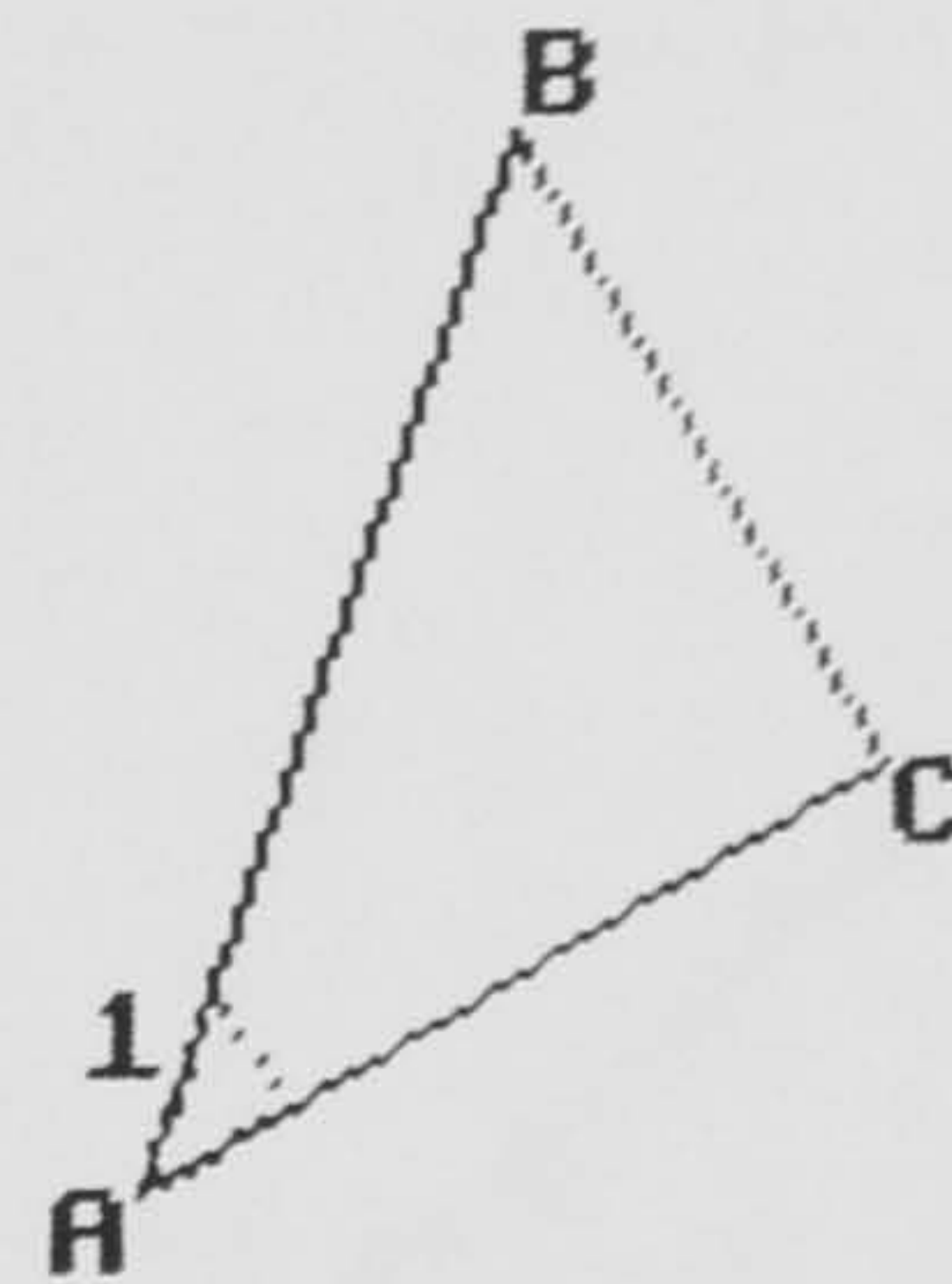


↑↓ select  
 & type  
 Display  
 F0-4 d.p.

AB=5  
 BC=3.21  
 CA=3.83  
 ∠A=40°  
 + ∠B=50°  
 ∠C=90°

+-\* / change

@ 130°  
 Rescale Un  
 ↔ stretch  
 Unit:h End



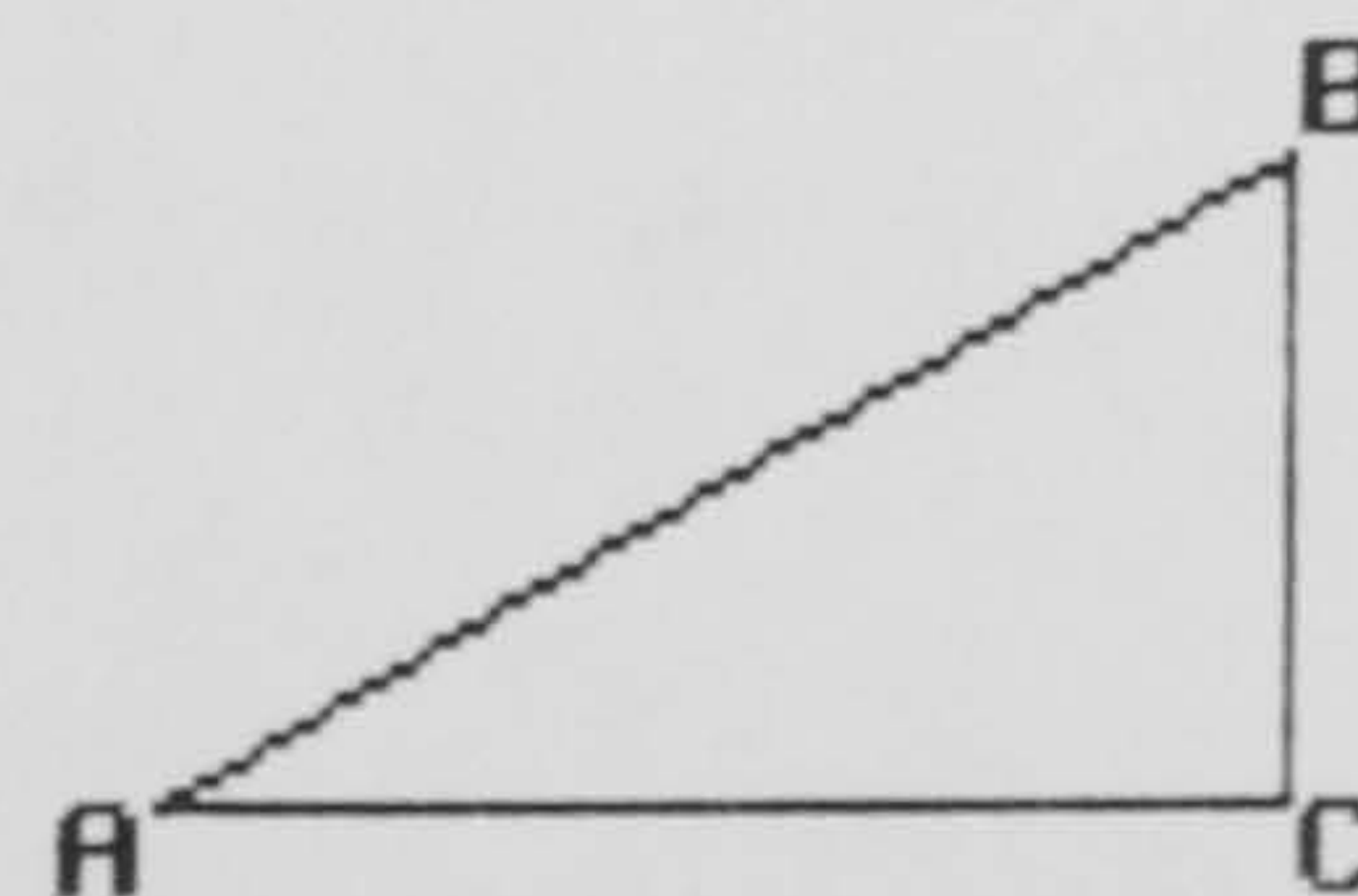
The Following two screen prints show an initial drawing where AB was inputted at 5 units, with angle A 30 degrees, and the numerical values revealed, then the user has given the instruction to add 10 to angle A. The second screen is then produced after pressing the return key.

↑↓ select  
 & type  
 Display  
 F0-4 d.p.

AB=5  
 BC=2.5  
 CA=4.33  
 + ∠A=30°  
 ∠B=60°  
 ∠C=90°

+-\* / RETURN  
 +10

@ 10°  
 Rescale Un  
 ↔ stretch  
 Unit End



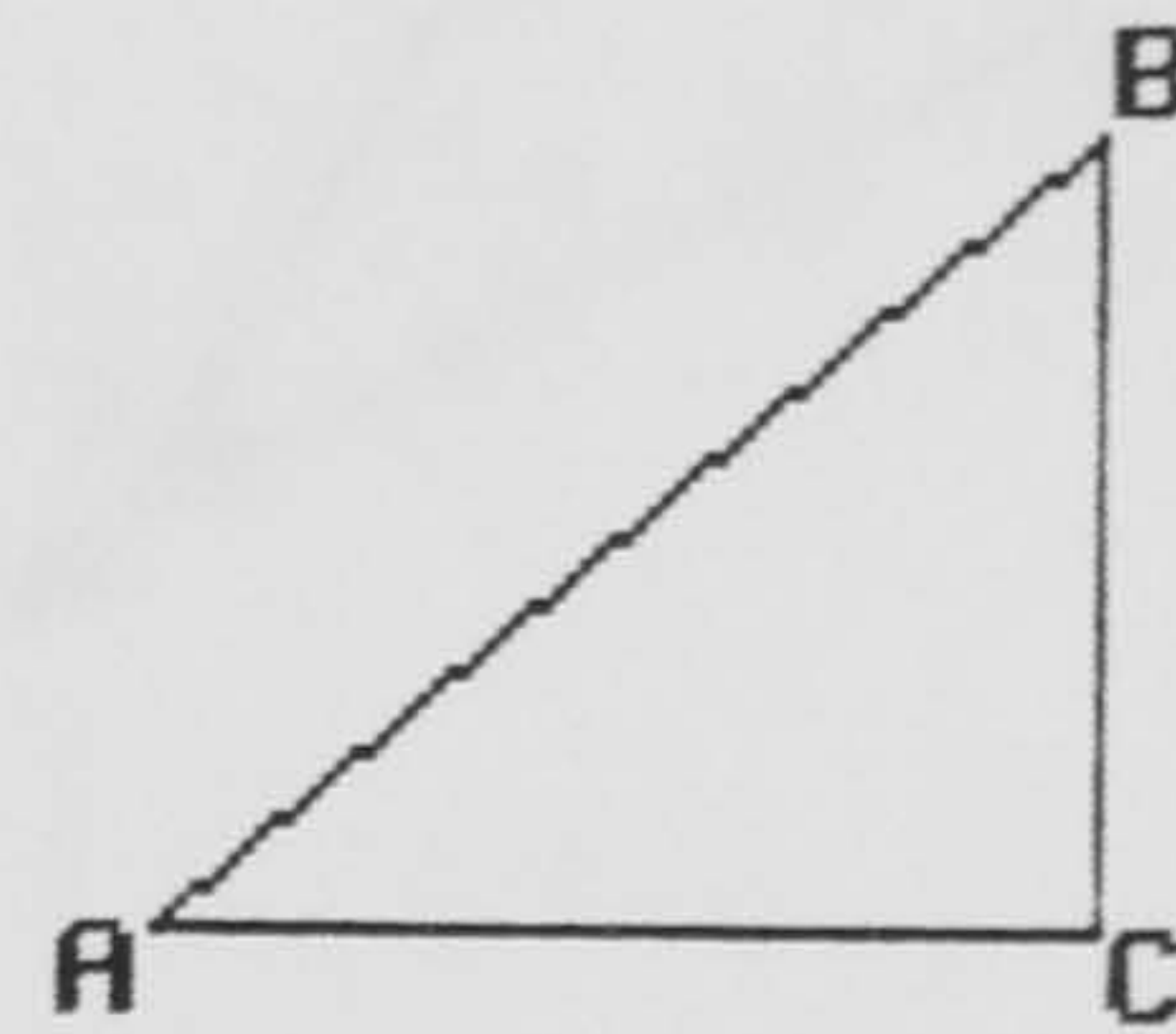


↑↓ select  
 & type  
 Display  
 f0-4 d.p.

AB=5  
 BC=3.21  
 CA=3.83  
 →∠A=40°  
 ∠B=50°  
 ∠C=90°

+ - \* / RETURN  
 +10

[0] 10°  
 Rescale On  
 ↔ stretch  
 Unit End



The drawing shows the changed values of the sides, which could be estimated and compared with the displayed values. Touching the return key would again add 10 degrees to angle A and redraw the triangle.

The next screen print shows a triangle constructed from an initial input of AB=10 and angle A=30 degrees, with the triangle rotated through 30 degrees and the unit hypotenuse marked, (with the construction of a line parallel to BC, to provide a unit triangle). The user has given the instruction to enlarge AB by a factor of 2.

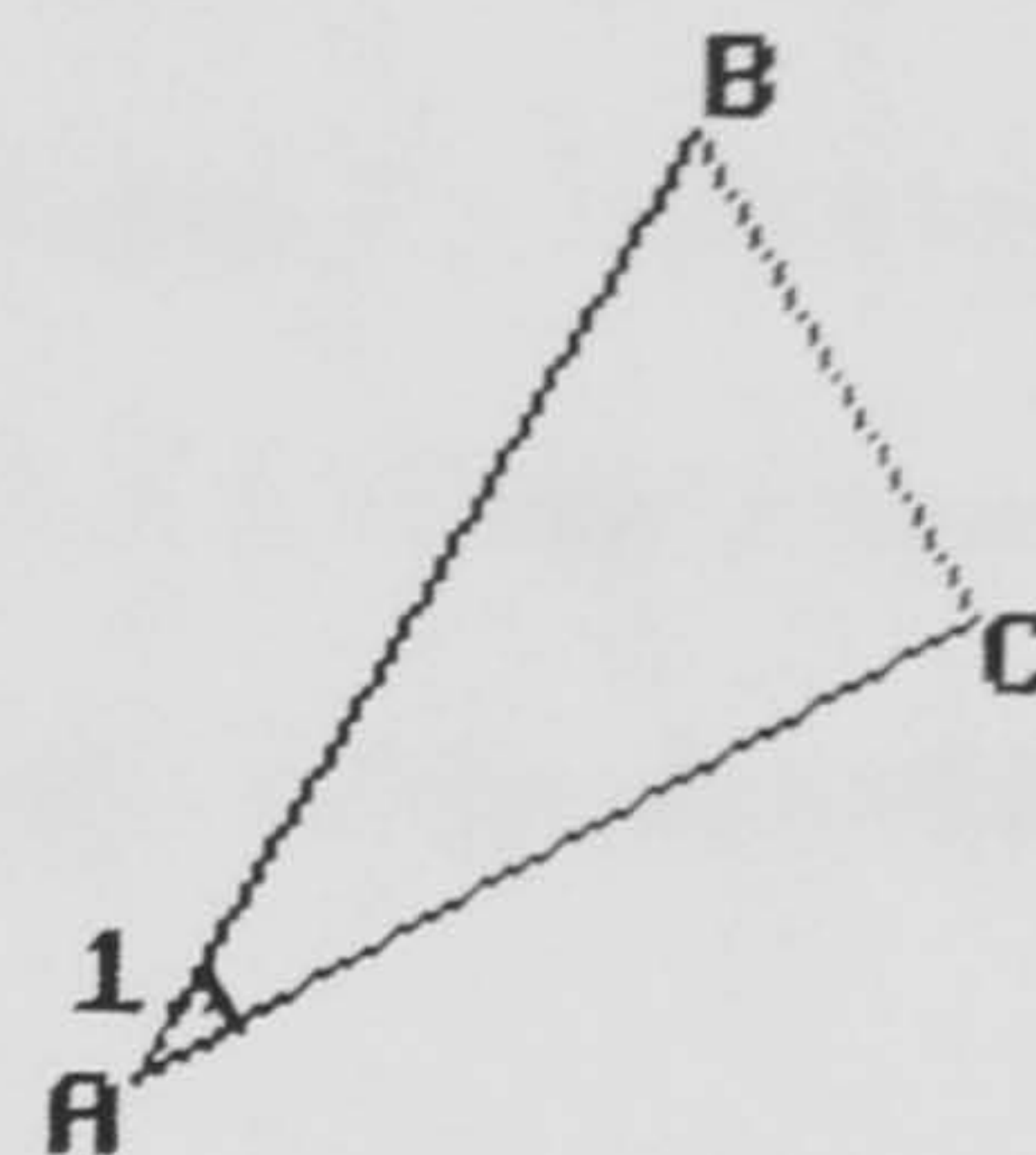


↑↓ select  
& type  
Display  
f0-4 d.p.

→AB=10  
BC=5  
CA=8.66  
∠A=30°  
∠B=60°  
∠C=90°

+ - \* / RETURN  
\*2

Ⓢ 130°  
Rescale On  
↔ stretch  
Unit:h End



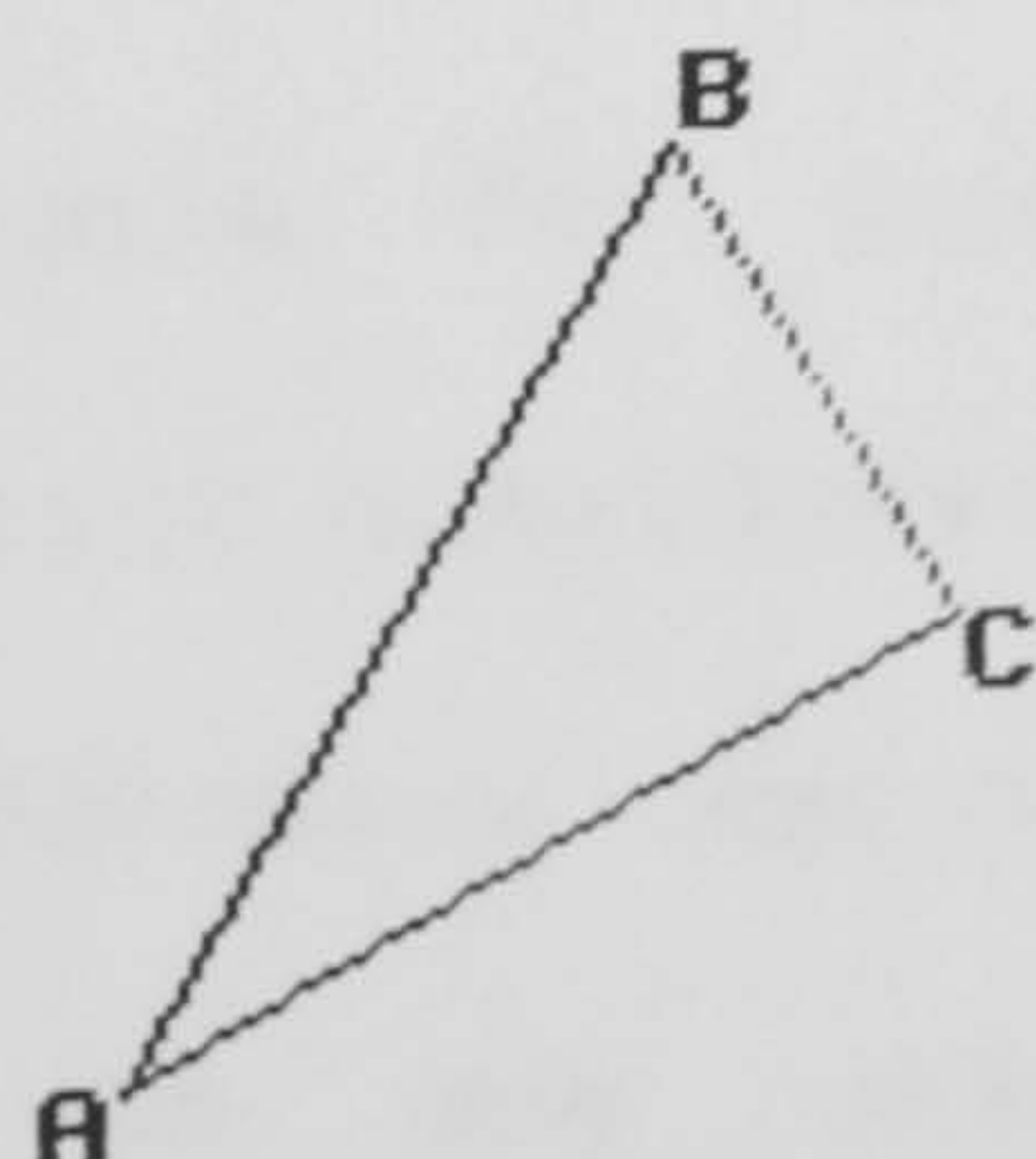
The resulting drawing does not 'fit' the screen, but the rescaling option, brings the triangle back onto the screen. The user now has to deal with the transformation of the 'unit ' value, seeing AB as 20 units.

↑↓ select  
& type  
Display  
f0-4 d.p.

→AB=20  
BC=10  
CA=17.32  
∠A=30°  
∠B=60°  
∠C=90°

+ - \* / RETURN  
\*2

Ⓢ 130°  
Rescale On  
↔ stretch  
Unit End





It can be seen that the computer program does not provide instruction, but facilitates learning. The program could only be used effectively within an overall strategy, involving exercises to be completed which would complement work carried out on the computer, and some teacher intervention.

#### **7.4 Pedagogical approach for the experimental groups.**

The teaching strategy to be employed with the experimental groups was designed to incorporate several techniques to promote effective learning. The first of these is that of group activities around the computer. In part this was necessitated by the limited number of computers available, but the main reasons for adopting this form of activity were as follows. It was recognised that importance should be attached to the pupils own language development as a means of guiding action and interpreting phenomena; group work can allow children a way of developing their thinking by giving opportunities for them to express their views to their peers. The importance of the social group in forming the goals for learning of the individual child was another important consideration, following Schmuck's argument (1978) that cooperative learning can provide valuable support to children who may otherwise become overdependent upon teacher direction. With an emphasis on group and paired work, relating estimating, sketching and calculating to computer work, it was thought that the search for non-meaningful procedures would be allayed and that anxiety created

by a surplus of teacher direction and supervision may be reduced. A further consideration was the possibility that the cognitive development of the children could be aided by discussion taking place when solving problems in groups or pairs. It has been suggested by Perret-Clermont(1980) amongst others, that this discussion could provide a mechanism for cognitive conflict, which would promote individual conceptual development. Such conflict is likely to occur in the use of this software when dealing with the rescaling of triangles and the range which the sine, cosine and tangent can take.

Papert's (1980) claim that children being fed information rather than playing an active part in the process are being denied the opportunity to learn in the way that Piagetian theory describes, and his emphasis on self-discovery and 'learning by doing' makes a valuable contribution to the way in which our understanding of pedagogy is evolving. However, The structured approach, advocated by Howe as the optimal learning environment, is very persuasive (Howe J.A.M.,1979). He suggests that the novice learner requires an hierarchical, model building, approach, with the model builder gaining familiarity with the building components and the assembly operations. The student should be able to move from one level of difficulty to another but help must be given by the teaching system in the form of explanations and examples, providing the 'conversion kits' necessary to move up the heirarchical structure. Whereas Papert beleives that the general rule will result from children recognising the similarity between



problems when involved in purely problem solving activities, Howe argues that even in LOGO this is very unlikely to happen, for the novice student is unlikely to have the extensive domain specific knowledge for this to occur. Howe believes that not to provide basic knowledge is to seriously disadvantage the learner by refusing them the most important information in the process of forming generalisations. Howe's observations should be placed in an information processing model and related to the work of Skemp and Ausubel, for the knowledge he suggests will be valuable, will only be so if it is capable of being linked to existing knowledge representation structures.

In terms of the information processing viewpoint, it is desirable that children will be able to form the concepts from recognising the perceptual patterns and to be able to visualise constructions as a result of earlier perceptions, but the specific patterns are unlikely to occur unless the exercises are carefully structured and the results are placed in a wider knowledge representation structure. The teacher can play an active part by linking exercises, providing demonstrations and explanations, and providing 'advance organisers' as described by Ausubel, for other stages of the work. The teaching strategy was therefore based on linking carefully constructed exercises with the numerical and visual imagery produced by the software, with teacher intervention and instruction designed to allow children to develop the 'conversion kits' necessary to link the various components of trigonometry to the overall global model. In this way the teaching of the experimental groups was in the

mode of Tall's (1986a) 'enhanced Socratic' mode. The software, as described above, was not itself a didactic package, but a facilitating program which would allow children to develop concepts after performing tasks at the computer. These tasks would not, for the large part, be self evident but would be suggested by the teacher as part of the overall teaching plan, stepping up levels of difficulty so that new concepts are subsumed into existing frames. However, the students would be working at the computer, either individually or in groups or pairs, and the way in which the exercises could be developed would allow students to develop concepts by discovering relationships presented in a visual and numerical form on the computer screen.

### **7.5 The Teaching of the control groups**

The following descriptions of the teaching of the control groups can only be seen in terms of a subjective approach with care being taken to involve the teachers concerned in discussions about the lessons they gave and the responses of the pupils. All four of the teachers concerned were asked to make written comments on the researcher's initial written draft which supplemented the ongoing discussions

#### **7.5.1 Set C1**



The introductory lesson was observed by the researcher and discussions about the group took place several times over the two week teaching session. The introductory lesson itself was heavily teacher based. Mr.A. drew a large triangle on the board, labelled the sides as 'opposite', 'adjacent' and 'hypotenuse' with the relevant angles marked. He gave the students the information that the hypotenuse was the name of the side opposite the right angle and that the side opposite one of the other angles was simply named the 'opposite' side. The third side was given the name 'adjacent'. Using these three names he went on to introduce the sine of the angle, saying it was just a name for the value found by dividing the opposite side by the hypotenuse. In a similar way he introduced the definition of the cosine of the angle, found when the adjacent side is divided by the hypotenuse.

His introduction was apparently listened to with interest and the teacher went on to suggest that they draw a triangle with a hypotenuse of 10cm. and an angle of 30 degrees and use this to discover the sine and cosine of 30 degrees. The pupils set about this task energetically producing the required response. Mr.A.went on to ask the students to try the exercise again using a hypotenuse of 5cm. to show that the same value, allowing for errors in measuring resulted from the calculation.

The teacher went on to give the students instructions of how to use their calculators to find the sine and cosine of 30 degrees to check the values found by drawing. The final stage of this introductory lesson was taken up with Mr.A. assuring



the students that the values of the sine and cosine of any angle could be found from the calculator and setting a short exercise on using the calculator to find sines and cosines of given angles, rounding the answers to three decimal places. The students appeared to have no problems with this exercise, enjoying the use of the calculator and the ease with which they could produce answers. They appeared to be quite satisfied with the lesson. The students had been keen to compare answers found by drawing the 30 degree triangle and, from their comments, many were of the opinion that trigonometry was beginning to make sense to them.

After the lesson Mr.A. spoke of the way he intended to develop the theme. He intended to introduce the tangent of the angle as the 'opposite over the adjacent' and suggest that the students try to remember the three ratios by learning a mnemonic. The pupil text book emphasised the ratios and the mnemonic. He would then try to show how the initial equations  $\sin A = \text{opp}/\text{hypotenuse}$ ,  $\cos A = \text{adj}/\text{hypotenuse}$  and  $\tan A = \text{opp}/\text{adjacent}$  could be used to find missing sides of the triangle. This would involve the algebraic process of multiplying both sides of the equation by the denominator of the fraction produced by the equation. From past experience he believed that this could cause problems for some students, but that as this was a very bright set of pupils he did not anticipate a great many problems. He did intend to allow the students to practise this technique however and expected two lessons to be taken up with this process. His next step in the planning was to use the same

equations to find the value of the angle. Again, he intended to spend some time allowing the students to practise the setting out of the equations and using the inverse buttons on the calculator.

After working in this way, Mr.A. intended to spend some time discussing the way in which the sine of the angle increased and the cosine decreased as the angle was increased from 0 to 90 degrees, before introducing problems from the text book relating to missing sides and angles of the triangle. He was aware of certain algebraic processes which he expected the pupils to find problematic. In particular the expression  $\text{sine } A = \frac{\text{opposite}}{\text{hypotenuse}}$ , when the hypotenuse was unknown.

After the first week Mr.A. felt that the pupils in this set had mastered most of the techniques he had intended to teach and was confident that they had achieved a sound understanding of trigonometry. The remaining lessons would be spent practising problems from the text book. He believed that the students would be ready for the first post-test in the later stages of the second week but would be prepared to allow the students the rest of the week to consider a variety of problems. He had been impressed with the way this set had worked through the exercises he had set, both on worksheets and from the pupil text-book, and felt confident that many were already well prepared for the kind of question they would need to master for the G.C.S.E. higher level paper. He believed that if the pre-test were given to this class again it would appear trivial.



Mr.A. referred to the group as "very bright" and "very quick, particularly some of the boys". He felt that his teaching style, of general didactic input followed by exercises from the book or from the blackboard "suited the class very well". Once the class were working on the exercises his role was to move from desk to desk whenever help was requested.

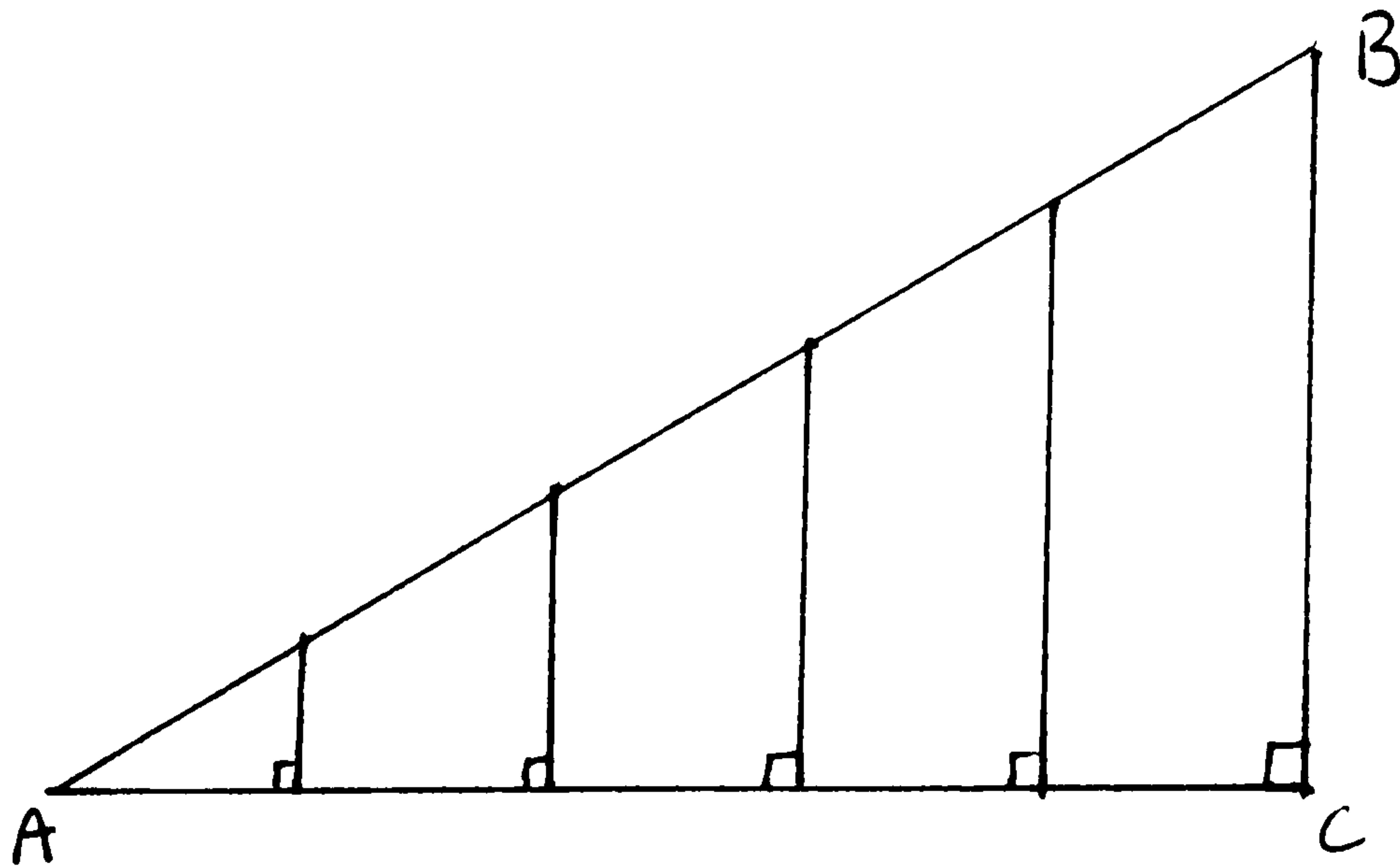
#### 7.5.2. The teaching of C2

The introductory lesson given to C2 was somewhat different to that given to C1. Mr. B. believed that the sine and cosine of an angle should be approached from the point of view of an appreciation of the properties of similar triangles. His initial statement that the class was now about to look again at trigonometry was generally well received, with some pupils openly stating that they had not understood the work they had encountered in the previous year. Mr. B. had stated that in his opinion most of the members of this class were well motivated and keen to be seen as able pupils, and the willingness with which they approached the task appeared to support this view.

The class were asked to draw a right angled triangle ABC with an angle of thirty degrees at A, and a right angle at C, with the hypotenuse such that the drawing filled the whole of one page of the exercise book. Then by marking off sections of the hypotenuse and drawing lines parallel to BC, as in the drawing below, they were asked to consider the results obtained by measuring each of the lines parallel to BC and dividing this



by the new length of the hypotenuse. Calculators were used to carry out the division processes.



Having established the result that these ratios were the same (approximately), Mr. B. went on to state that for the angle of thirty degrees, this ratio was called the sine of thirty degrees. He asked the students to input 30, then press the sine button on the calculator to confirm that the calculator would produce that value they had judged to be the sine of thirty degrees. He asked the class to construct a right angled triangle with an angle of forty degrees and use this to find an approximation to the sine of forty degrees, using the calculator to check the answer.

The cosine of thirty degrees and forty degrees was found by measuring and dividing, using the calculator to compare answers. Finally, the statement "sine = opposite/hypotenuse,

cosine =adjacent /hypotenuse" were recorded in the pupils' exercise books.

Five pupils, three boys, two girls, were asked to discuss the lesson with Mr. B and the researcher at the end of the period. There was a consensus that they had "understood "the equations they had written down but were unsure why they had been required to draw the triangles and measure the sides. It would appear that the final equations provided the same reassurance that was felt by members of C1, that they would be able to complete calculations successfully; the initial 'discovery' was not considered particularly relevant.

Mr B. was to proceed with a series of lessons very similar in style and content to those given by Mr.A. with C1, concentrating on performing the operations necessary to calculate the missing sides or angles. When asked if the pupils in this class had experienced any difficulties, he was able to identify certain algebraic skills which he felt were not fully 'understood'. Foremost of these was the predicted problem of finding the hypotenuse when one of the shorter sides and an angle were given, requiring an extra line of algebra. In addition he was troubled by the way in which some pupils were using the calculator without writing down equations, or becoming confused between the conventional notation,  $\sin 30$ , and the order of inputting the calculator which would be expressed as  $30 \sin$ . He also felt that, as in the past, pupils found it very difficult to associate the exercises and exposition he would provide in order to establish maximum and

minimum values of the sine, cosine and tangent for angles between 0 degrees and 90 degrees, with the trigonometric calculations being carried out at other stages of the programme. However, he felt that these pupils now had a much clearer understanding of trigonometry and would be able to perform much better than they did on the pre-test.

Both Mr.A. and Mr.B. had used the pupil text book extensively, practising routine questions given in the form of diagrams with sides or angles to be calculated and 'problems' given in sentence form. It was thought appropriate, by both teachers, that any problems given in the post-test should be similar to those in the text-books.

### 7.5.3. The teaching of C3

The initial teaching of this class appeared to the observer to be very different to that of C1 and C2. As they had already had a number of revision lessons before the pre-test the introductory lesson after the pre-test was in fact repeating information they had received only two weeks earlier. The pupils were clearly very concerned about their apparent inability to "do trigonometry" and were very much more reticent about having more explanation given to them. However, the technique employed by Mr.C. was in direct response to the demand for the equations to be written upon the board with the mnemonic which had clearly failed them in the past.



Mr.C. was concerned to reassure the pupils that if they remembered the mnemonic and knew how to use the calculator, there would be little problem in solving the questions he would ask of them. In a similar manner to Mr.A. with C1, he drew the triangle on the board, carefully wrote the equations for  $\sin x$ ,  $\cos x$  and  $\tan x$  in terms of the opposite, adjacent and hypotenuse and the mnemonic SOHCAHTOA. The pupils wrote this down, though they had written the same information in the same form during their revision periods, and were asked to use it to solve simple questions. An example was given on the blackboard and the pupils were asked to work through this example on the calculator together. The question was not placed in a specific context, but the hypotenuse was given in terms of a length in centimetres.

Unlike the other two sets, there was a need to have calculators available, for the majority of the members of this group had not purchased their own calculators. Upon arriving at the correct answer, much of the initial doubt was dissipated and the pupils were anxious to try other questions. However, it was noticeable that there was a great deal of reluctance to write down any representation of the calculation performed in their exercise books; most wrote only the question and a numerical answer, some put only a number, ignoring the units given in the question. The pupils were working in groups of three or four rather than the pairs which were evident in the other two sets and there was in general more peer questioning

about how answers were being calculated, and disputes about results, than was evident in the other two sets.

After the lesson Mr. C. expressed the view that the class had really had enough of drawing and measuring triangles, (he had already used the drawing and measuring of similar triangles with this group) and that the only way in which he could succeed in teaching trigonometry was to rely on their ability to memorise the routines. He felt that for most of the class trigonometry was simply too difficult, but that if they experienced success in following routines, using the calculator successfully, then the pupils were likely to be able solve the more basic trigonometry questions associated with the intermediate papers of the G.C.S.E..

He too was planning to use the text-book extensively, but would only consider the easier questions and problems based on simple situations. He believed that the members of this group would be able to write some algebraic statements representing the trigonometrical ratios, based on the mnemonic, but would be unable to express stages of working in an algebraic way. He imagined that the class would practise techniques for a week or so but that it would be unwise to continue with this topic for longer than this.

At the end of the teaching programme Mr.C. was confident that he had taught the class to perform the routines necessary to answer the more straightforward questions but that he had not been able to extend these skills. He believed that to go on teaching trigonometry would have been counter productive in



that the class were finding the more difficult questions disheartening.

#### 7.5.4. Commonalities in the approach to the teaching of C1,2,3

Interestingly, since all three teachers were involved in the teaching of third year groups, none of the teacher favoured the approach being taught to the third year pupils. The unit hypotenuse, with the sine and cosine of the angles being defined in terms of the length of the opposite and adjacent sides had been the main part of the pupils work in the third year. It would seem that the teachers were persuaded that this method was the current view of how trigonometry should be taught, but that for most pupils it was unlikely to be successful. In fact it was considered by all three teachers that the unit hypotenuse method had been meaningless to pupils of average ability, some of whom, when finding they did not understand the work, were being retaught by parents or private tutors using the ratio definitions.

All three of the teachers had been in favour of 'setting' pupils in the fourth year, yet the lessons given to each of these three groups were very similar in style and expectation. A major difference, however, was the differing amounts of confidence being placed on whether the pupils had much to gain from studying trigonometry at all. Although there was a selection of text-books available for fourth year pupils, particularly the Oxford Comprehensive series, which purport to



differentiate material for differing ability ranges, all three teachers chose to base the questions given to the pupils, and the style of presentation of results, on the same text-book. The reason for this was that the three teachers considered the learning of the trigonometric equations was the basis of understanding. After this the pupils needed to practise the routines involved in applying these equations to broadly similar problems; the text-book chosen, (Raynor D,1984), is described as a revision book, giving only the barest of information before supplying a great number of practice questions. The trigonometry section contains only the following information, before two examples on how to find missing sides then an exercise containing thirty-four examples:

Three important functions are defined as follows:

$$\sin x = O/H$$

$$\cos x = A/H$$

$$\tan x = O/A$$

It is important to get the letters in the right order. Some people find a simple sentence helpful where the first letters of each word describe sine, cosine or tangent and Hypotenuse, Opposite and Adjacent.

An example is:

*Silly old Harry Caught a Herring Trawling  
Off Afghanistan*

e.g. S O H:  $\sin = O/H$

(P 148)

#### 7.5.5. The teaching of C4

This group was viewed in a different way from the other three control groups in that it contained fewer pupils (16) and these

pupils were following a course which did not contain any trigonometry. It was thought unwise to give these pupils a pre-test as they would be unlikely, in the light of the other pre-test results, to show any real understanding of trigonometry. Indeed, many of these students would have been given other work to complete when the other third year students were attempting to follow the third-year trigonometry programme. However, their teacher, Mr.D, felt that with such a small group it would be interesting to approach the subject from a practical geometric viewpoint in order to see if some or all of the students would be able to gain an understanding of the basic trigonometric functions and their application to practical problems.

In the initial lesson Mr.D. simply encouraged the students to draw right angled triangles given basic pieces of information, three sides or two sides and one other angle, then asked them to measure the other sides and angles. To this extent he followed very much the holistic geometric approach employed in the experimental groups. This double lesson was observed by the researcher.

It appeared to be very successful with all of the pupils taking care to produce scale drawings which they were able to use successfully in measuring the missing dimensions. The students were certainly diligent and able to work well in pairs or groups while Mr.D spent time helping those who were unable to interpret the instructions he was giving. The drawings they were asked to produce were in fact scale drawings of diagrams drawn by Mr.D on the blackboard, but there were no observations



made by then students when the drawings they produced did not look like the diagrams in terms of relative sizes of sides or angles.

Mr. D. was to spend two weeks developing this idea, using measuring and dividing of sides to build up a table of ratios for angles between 0 and 90 degrees, before introducing the names 'sine' and 'cosine'. He only introduced the calculators at a late stage of the teaching to show how the tables they had produced could be produced on the calculator by pressing the appropriate function keys. Mr.D was able to supply all of the calculators necessary, for it was noticeable that the proportion of each group who had purchased their own calculators, in line with the policy of the school, declined in line with the purported ability of the teaching groups.

When he saw the post-test, Mr.D. felt that the students would have some difficulties but he expected them to show some understanding of sine cosine and tangent in the first section and be able to attempt then simpler questions in the second stage. He had not rehearsed the group through many practice questions, which had been a feature of the teaching of the other three control groups, but he had given them some standard problems to solve, mostly from then same text-book used with the other groups. He felt that he had been successful enough with this teaching strategy to suggest that many of the group could attempt the intermediate G.C.S.E papers rather than the lower standard.



The teaching style of Mr.D, with a group of this size, was noticeably relaxed, with the pupils reluctant to apply the formalities associated with large group teaching. When they required the attention of the teacher they tended to call for it; they were clearly much more accustomed to leaving the desk and walking around to see what others had produced; they were able to intersperse talking about the mathematics with more general unrelated conversation. In general, much of the teaching could be seen as being similar to that intended for the equivalent experimental group, except for the use of the computer. In this sense the results obtained by the two groups can be seen as vital in considering the major hypothesis of the research.

#### **7.6 The teaching of the experimental groups**

The experimental groups were taught by the researcher roughly simultaneously to the teaching of the equivalent control groups, with the post-tests given to each equivalent group approximately two days after the teaching sessions had been completed. The researcher had not met any of the groups prior to the commencement of the teaching, so the teacher/pupil familiarity, which was evident in the teaching of all of the control groups was an inevitable difference between the experimental and control group teaching. In order to accommodate the computers in a suitable teaching room and move them at the end of each teaching session to safe storage

facilities, the four experimental groups were taught in the same classroom. This in itself was a disruption to established routine and was the source of some confusion at the beginning of each of the teaching sessions.

The classroom used contained large tables which were arranged in columns and rows with the intention that two pupils would sit at each one, though they were considerably bigger than two traditional school desks. Only one electric socket was available and this was situated by the side of the blackboard at the front of the room. The room was spacious, allowing for rearrangement of the tables to accommodate the three computers yet still able to seat the pupils easily.

The teachers of the equivalent control groups were invited to attend the lessons and make any comments on the teaching they observed: all four of these teachers attended at least one of the lessons.

#### 7.6.1. The Teaching of E1

Lessons 1 and 2, 2 thirty-five minute periods.

The initial exposition was designed to explain that we would be concerned with sides and angles of triangles, in the first instance right-angled triangles. Mention was made of trig. points in surveying and the kind of problem associated with triangles. The computer program was introduced as a device which could draw right-angled triangles quickly and provide the dimensions of sides and angles not specified. Emphasis was



place on the ability to make reasonable estimates of missing sides by either producing a scale drawing, or by drawing a sketch where the angle drawn was reasonable close to that specified in the original data. A sketch of a right-angled triangle ABC was given on the board, where the hypotenuse AB was marked as 10 units and the angle A was marked as 60 degrees. The class were asked to consider what kind of estimate could be given for the sides AC and BC. Suggested answers, from two of the girls were given back to the class to gauge the level of agreement. Clearly this presented a problem to some of the pupils. Two of the girls, sitting together, expressed a sense of confusion at the question, not recognising how other members of the group were able to make their estimates.

At this stage the class were divided into groups and asked to sit around one of the three computers whilst one of the group inputted the data necessary to produce the drawing of the blackboard problem on the screen. The unit on the hypotenuse was marked by using the appropriate input, and the groups were asked to consider how many of these units would fit into the sides AC and BC. The speed at which this process was carried out and the signs of recognition on the faces of some of the members of each group gave the indication that this method supplied a breakthrough to the estimation process. The two girls who had earlier expressed confusion were now able to see how to proceed. This was a vivid example of the ability to transfer visual images from the screen and use them to produce mental imagery, for the students did not have to draw units and



step them out on their own constructions in order to produce the estimates but were able to perform a mental version of the same process.

This largely teacher led exposition had taken the first of the periods and part of the second period. For the remainder of the session the students were working in groups from a work sheet which asked them to draw sketches of triangles, ABC each with hypotenuse AB of 10cm, and asked them to estimate the sides AC and BC, then to use the computer to check the answers. This activity caused some concern amongst many of the group; the notion of estimation given as a vocal response appeared to become a much more serious issue when the response was to be given in a written form. This was the first indication of a reluctance to abandon the 'routine' thinking which had obviously been part, of the success in achieving the position of 'top-set pupils'. Two of the boys thought that as the computer would give the answer anyway the exercise had little value, and several students expressed a concern that they should have a way of establishing a correct answer before they wrote it down.

The process of estimation, where an opinion was sought and then the accuracy checked, made some students feel a little nervous. However, after some attempts were made, and the pupils, working in groups, drew the triangles and checked the results, there were many more positive feelings expressed. Some were surprised that they had made such good estimates despite the fact that they had not even used a protractor to measure

the angles. At the end of the session a group of girls who had been eminently successful at the exercise expressed their pleasure at their success but were concerned that the computer program did not appear to have much to tell them about trigonometry, "those sins(sic) and tangent things".

#### Lessons 3 and 4

The next session was devoted to establishing a table of sines and cosines of angles from 0 degrees to 90 degrees, based on estimating the sides of the triangles drawn on the computer screen. The terminology 'sine' and 'cosine' was not used initially, but the sides AC and BC estimated visually from the drawing and compared with the numerical values revealed on the screen, when the hypotenuse was fixed as 1 unit. The 'adding-on' facility which allowed the angle to be increased by ten degrees at a time, was encouraged. Again the students worked in groups, filling in the elements in a table drawn in their exercise books. This exercise proved to be very successful, taking a very short time for the students to establish some important ideas, without any teacher intervention at all. The estimates were given to one decimal place and on the whole were remarkable accurate, the actual values were given to four places of decimals when revealed on the screen, in the column to the left of the drawing.

The groups quickly perceived that one side was increasing as the other decreased and that there seemed to be a pattern in the values they were writing down. By the time they were inputting 70 degrees for angle A many were correctly predicting



the correct values for AC and BC from the values already established for the input of 20 degrees for angle A.

Inevitably, questions about rounding values to a given number of decimal places occurred when the revealed values were discussed in relation to the estimated values.

The teaching input, following this exercise, was to ask questions of the whole class based on consolidating the findings from the group work. In particular, the value of angle A which would produce the same values for AC and BC and an estimate of this value. Some discussion was also necessary about the legitimacy of using 0 degrees and 90 degrees for values of angle A. In addition the use of the words 'opposite' and 'adjacent' were introduced to allow questions involving a change of focus of our estimates from angle A to angle B and to introduce problems associated with differing orientations of the triangle.

The later stages of the session were designed to allow the class to appreciate the difficulty involved in estimating sides visually from blackboard diagrams, where the angles given were inaccurately drawn. By drawing their attention to the uses of diagrams as a symbolic way of transmitting information about the triangle, without being necessarily visually accurate it was hoped that they would be able to produce their own visualisations of the triangle and make reasoned inferences about dimensions. An exercise on estimating the sides of unit hypotenuse triangles in differing orientations (reflections and rotations of the standard position of the computer drawn



triangle) and angles of any size between 0 and 90 degrees, using their tables of values as a basis for the estimations consolidated the earlier work.

The final statements were intended to present an overview of the process so far and an advance organiser for the next session. The teacher advanced the idea that the ability to solve right angles triangle problems could be aided if the table of values, to varying degrees of accuracy, could be easily called upon and used for scaled-up or scaled-down triangles. A reminder of the kind of problem we were trying to solve was given in order to keep the global view in mind. The point was made that calculations based on these tables of values were crucial to the effective use of trigonometry but visualising the triangles could give a valuable insight to the relative sizes of the unknown dimensions.

Some girls were noticeably reluctant to use the computer keyboard, though they were happy to observe the activities, whilst two of the boys were keen to display their skill as computer buffs, rather than mathematicians. Though the session was generally pleasing there was still some sense of searching for a routine amongst at least two of the girls. They appeared unhappy about not yet establishing the procedures associated with trigonometry. They were aware that routines existed to solve trigonometrical problems because of previous tuition and help from parents, though they had not been at all successful in the pre-test, and appeared anxious to "learn" these methods.

**Lessons 5 and 6**

The next session, the final one of the first week, was to introduce the calculators as a means to supply the lengths of the opposite and adjacent sides for given angles, when the hypotenuse was one unit. The initial teacher input was to introduce the terminology of sine and cosine in these terms and when sitting around the computer to estimate lengths and compare displayed values with those achieved using the calculator. All but three of the students had bought calculators, the others were supplied with one from the school stock, and all of the pupils were soon able to perform the necessary operations. Questions of 'rounding' answers occurred again when comparisons between calculator answers and the previous table were made. The next didactic input was to point out to the class the way in which we, by convention, write, for instance, the sine of 10 degrees as 'sine 10', despite the fact that the operation performed on the calculator may suggest that it should be written as '10 sine'. The next thirty minutes was spent finding missing sides of triangles drawn on a work sheet in varying orientations, using the instructions that the students should:

1. Either redraw the triangles more accurately or by visualising the triangles drawn more accurately, decide which unknown side is the bigger and estimate the lengths.

2. Using the calculator and expressing the sides in terms of sine or cosine of the angle calculate a more accurate length for the two sides.



3. As an option they could use the computer to draw the triangle, reorientating it to fit into the standard diagram and check the results.

Most of the class carried out all three of these activities.

The second half of this session was intended to illustrate the enlargement of the unit hypotenuse by a given scale factor and the similar enlargement of the sine and cosine of the angle. Sitting around a computer in groups, with one student inputting the information at each of the computers, the hypotenuse was inputted as 2 units for a given angle of 20 degrees. The unit was marked on the hypotenuse and the groups identified the sine and cosine of twenty degrees on the drawing. Using the calculators they were able to calculate these values. They were then asked to consider how AC and BC related to these values and appeared to have no trouble in seeing that they had doubled. Then by multiplying the sine and cosine of twenty degrees on the calculator they were able to suggest what the sides AC and BC would be, confirming the answer by revealing the values on the computer screen. Using an angle of 40 degrees and a hypotenuse of 5 units the process was repeated, with very little difficulty. The groups inputted their own value for AB, between 1 and 5 units, and angle A, displaying the values of AC and BC as well as the angle B, then employed the scaling factor \*3 to see the effects on the sides and angles of enlarging in this way.

It only remained for the teacher to sum up this process with instructions about how it could be written in an algebraic



form ( $BC=5\sin 40^\circ$ ), before allowing the students to attempt a series of enlargement problems which had to be completed on the computer, working in pairs and written in the conventional manner in their exercise books and completed for homework.

One of the main values of this kind of exercise is that the students are either working in their exercise books, using calculators or working at the computer. No one is waiting with nothing to do.

By this stage the students had become familiar with the computer program and had gained a clear understanding, from the computer drawn images and numerical relationships of the meaning of sine and cosine of an angle and how to use these in order to find 'missing' sides in right angled triangle questions. Importantly, they were also able to estimate results from accurately drawn diagram or from visualisations of the triangles represented by the more symbolic diagrammatic forms.

#### **Lessons 7 and 8**

The next session was spent in looking at the problem of finding missing angles when all three of the sides of the triangle were known, basing the exposition on the visual images created by the computer diagrams. The teacher introduced the session by asking the class to sit around the computers and selected three students, to input the information into the computers. The problem itself was stated by the teacher, that is, we were about to examine how we could calculate the angles of a right angles triangle if we knew the value of two sides, using our knowledge about sines and cosines of angles.

He reminded the class about the definitions of sine of an angle and cosine of angle in terms of the unit hypotenuse before asking three students at the keyboards to input the values of AB as five units and AC as four units. He asked the class to note that the computer could draw the triangle with this information; it did not need to know the value of AC (the angle C is fixed at 90 degrees). The class were asked to estimate the size of angle A and suggestions were called for as to how angle B would be calculated if angle A was known. The three students at the keyboards were then asked to display the unit hypotenuse by pressing the appropriate key and the class was asked to consider scaling the original triangle down so that the hypotenuse was scaled from five units to one unit. They then had to consider what the size of AB would be after this scaling down process. They were invited to use their calculators to suggest the value of AB, for the process of dividing by five was clearly not something which some students could do mentally.

After arriving at the value of 0.8 we had to consider that we had now found the cosine of angle A because it was the adjacent side and we should be able to use this to find the angle itself. At this stage the teacher introduced the idea of the inverse cosine, in the sense of reversing the procedure they had used previously. The inverse cosine button was explained as a facility for inputting the length of the adjacent side when the hypotenuse is one unit, and the calculator finding the angle included by the hypotenuse and the



adjacent side. Using their calculators in this way each member of the class was able to arrive at the value of angle A in the computer diagram.

Having arrived at the value of angle A, the class were asked to find angle B in the quickest way and these were compared with the original estimates and the values revealed by the computer. The whole process was repeated for another triangle with a hypotenuse of eight units and the value of AC given as seven units. After the second computer based example the teacher reviewed the process on the blackboard introducing the algebraic statements based on  $\sin \text{ of angle} = \frac{\text{opposite}}{\text{hypotenuse}}$ ,  $\cos \text{ of angle} = \frac{\text{adjacent}}{\text{hypotenuse}}$ , linking what had been carried out in terms of the scaling down procedure to the statement made in their text books. The class was then asked to attempt an exercise based on finding angles. The teacher spent the rest of the session marking homework answers and responding to requests for help. Some pupils needed to review the process again, particularly the use of the inverse button on the calculator, but most were able to complete the questions set with little difficulty. Before the lesson finished the final ten minutes were spent reviewing the story so far, before the class were left to consider some problems, given in word form, based on finding missing sides or missing angles, for homework.

It was obvious that there was a need to establish formal methods of setting out work and helping the students to remember the processes to be carried out. It appeared that they



had established a good understanding, in global, visual, terms of what they were being asked to do. However, the efficient use of the algebraic, serial, processes needed some practice. In addition the number of individual items to remember were increasing.

The beginning of the next session, when the answers to homework problems were discussed seemed the appropriate time for introducing an aide de memoire for the information they had used so far. The mnemonic from the text book was shown as a commonly used method of remembering the definitions of sine and cosine, though it was emphasised that the form in which the information was stated was not the best way of dealing with the problem of finding missing sides, where it was easier to think in terms of enlarging the sine of the angle by the appropriate scale factor to find the opposite side, and the cosine by the same factor to find the adjacent side. However, it was shown that the definitions of sine and cosine given in ratio form could be simply transformed into the expressions used previously by multiplying both sides of the equation by the value of the hypotenuse.

For the next 30 minutes further problems were attempted, with the teacher working with groups of students. Difficulties arose primarily from interpreting questions and drawing appropriate diagrams though students appeared very responsive to visualisations of possible answers, often making very good estimates, when requested, of the possible solutions. The

computer drawings had given a means by which to visualise even when they were not using the computer.

The final part of the lesson consisted of the teacher introducing the problem of finding the hypotenuse of the triangle when one side and an angle were known. At this stage the visual scaling up or down approach could be combined with an algebraic approach, and both were used to explain the series of expressions representing the calculation. In the first instance the class were asked to examine the diagram on the blackboard from the point of view that the opposite side of length 5 units, as well as the hypotenuse had been scaled up by the factor  $L$ , the unknown length of the hypotenuse, from the unit hypotenuse triangle. It follows then that the length 5 must be the sine of  $A$  multiplied by  $L$ , written as  $5 = L \sin A$ . Finally by dividing both sides of the equation by  $\sin A$  we have  $L = 5 / \sin A$ . Though the first part of this explanation can be seen in visual terms the latter stage is dependent on the ability to switch to a serial based algebraic approach.

As expected there were several requests to show this process again but with few noticeable exceptions the class were able to calculate the answers to two questions, based on this technique before the end of the lesson.

#### Periods 11 and 12

It was left until the last double period session to introduce the idea of the tangent of the angle, though previously mention had been made of practical problems which produced known values of the opposite and the adjacent, but the



hypotenuse was not involved. The notion of tangent was introduced on the computer, sitting in groups, by specifying that we were looking at the values of BC for different values of angle A given that AC and not the hypotenuse was to be specified as one unit. A work sheet had been prepared asking the group to estimate from the drawings, and then reveal the values of BC for various angles of A, including 45 degrees, from 0 degrees to 80 degrees. This exercise was followed by reflecting on the values revealed, with students offering suggestions as to what could be considered as interesting or important results. It had clearly been evident that the tangent of the angle was not restricted to values between 0 and 1 unit, and that the tangent of 45 degrees was in fact one unit. The calculators were used to show how these values could be found and the scaling up process was shown in the same way as in previous lessons.

A further blackboard diagram was used to illustrate the use of the name 'tangent' by drawing the unit circle around AC and the use of the statement  $\text{tangent} = \text{opposite} / \text{adjacent}$  was shown in terms of scaling down the adjacent side to one unit. The class completed a series of questions, given in diagram form and problems given in word form before the end of the lesson, with some students using the computer to check results at frequent intervals. The teacher was moving from group to group, introducing the idea of switching angles if the opposite side instead of the adjacent side was given in the question and finally summing up this technique and summarising the important



ideas encountered in previous lessons. The students were encouraged to memorise the mnemonic SOHCAHTOA in order to help them remember the links between the names sine cosine and tangent and the ratios.

The teacher/ researcher made a note to the effect that this final session was rushed and more time should be spent on this with other classes. The post-test was to be given four days after this final teaching session.

Notes taken during the teaching sessions highlighted the fact the boys in the class appeared to be subdued, not seeking help, though responding to help when it was offered. There were some obviously very able girls in the set who had very little difficulty with any of the concepts introduced, and in general the students were able to move between the visual interpretation and the algebraic notation and method with surprising versatility. Mr A. attended the second and part of the final sessions, expressing interest in the approach being used, though a certain amount of surprise at the fact that the students were not able to deal with the tangent of the angle at this late stage of the teaching. However, he commented on the apparent skill in estimating results and in the way they were able to draw conclusions from the computer drawings.

#### **7.6.2 The teaching of E2**

The teaching of E2 was planned in a similar way to that of E1 though it was unclear how much of a difference there would be

between the majority of the members in each set. The programme itself followed a similar pattern to that outlined above so it would seem appropriate to consider points which illustrate differences or similarities between the responses of the two groups.

The most striking point at the beginning of the teaching sessions was the difference between the attitude and approach to the lessons displayed by the majority of the boys in this group and those in E1. Notes taken at the time reveal that much of the teachers attention was taken up in responding to very vocal demands from some of the boys. It was clear that many had become so involved in the association of mathematics with procedures that they were impatient with a more conceptual approach. When the computers were used the same six boys intended to dominate their use, physically moving other away from the keyboards. Rules of behaviour had to be established, though their enforcement impinged on the teaching time. Similarly, the use of calculators caused more of a problem, with several pupils, notably the more vocal boys, not bringing calculators to the lesson so requiring the use of those stocked by the school.

The initial visual introduction and estimation proved to be successful, with the pupils displaying considerable skill in estimating, though problems arose when dealing with sides of less than one unit, for a poor knowledge of decimal notation and unclear understanding of numbers less than one proved to be a handicap for some of the class.



The pace of the lessons had to be somewhat slower and the need to consolidate learning with supervised practice became more evident. At the point where the students were dealing with scaling down triangles to find missing angles some of the boys who had previously proved to be reluctant to breakaway from a search for routines had reached the point where they were unable to think about the problems clearly. Four of the boys, sitting in one group, had seized the opportunity of using calculator routines to solve the problems, to the extent that they were unwilling to attempt a more visual approach.

Unfortunately they were vocal in their requests for attention in order to verify, or not, the operations they were carrying out. "Do you cos it now?", a question from one of these characters, illustrates the search for calculator routines in order to answer questions based on a supposed understanding of using the inverse cosine to find a missing angle. However, the other members of the class seemed to be capable of following the same teaching programme as that used with E1 with success. Where problems did arise they were more likely to be linked to difficulties in interpreting word problems in order to arrive at suitable diagrams.

### 7.6.3. The teaching of E3

Interestingly the notes taken of the teaching of this group, indicate that there was no perceived difference in the way in which they were able to understand this work from the members



of E2, or for that matter many of those in E1. The interpretations of the computer graphics, including the fact that  $\sin A = \cos(90-A)$ , was completed in the same investigatory fashion and the ability to estimate solutions was impressive. There was no cabal of demanding boys to dominate teacher activity and the general enthusiasm for the work was pleasing.

A point of difference which became more evident as the programme developed was the reluctance amongst many of this group to write anything down except the question and the answer. As with many of the members of E2, the difference between the written statements and the order of calculator operation was a source of confusion and concern. Even after copying examples the formalism of trigonometric equations was often replaced by pictures of calculator buttons in the order in which they were pressed. However, the visual approach allowed them to progress to a level where they were attempting and solving problems of a greater complexity than had been envisaged by either the researcher or their class teacher.

#### 7.6.4 The teaching of E4

The composition of E4, six boys and six girls, and the expectation of the teachers that they were likely to find trigonometry beyond their grasp naturally made the teaching of this group somewhat different from the other three experimental groups. The observation of C4, the corresponding

control group, which had taken place two days previously, had to some extent indicated the difference in expectation between these groups and the others but it was unclear how much of the teaching programme they would be able to follow. It had been made clear by their teacher, and the head department, that this group would not normally be expected to study trigonometry, as they were likely to be aiming for the lowest level of G.C.S.E paper if indeed they were to sit a terminal examination at all.

The small number of students allowed the teacher to use the computers very effectively, using groups of four or pairs, and the students were able to make good use of teacher time by asking for help at regular intervals. The most noticeable aspect of the teaching of this group, which supports earlier findings in similar research (Blackett N, 1987), was that the computer graphics environment and the visual approach had a remarkably stimulating effect on some members of the group. The enthusiasm and pleasure shown by a significant number of this group was a reflection of the success they were experiencing. They were able to follow a similar teaching programme to that given to the other groups with perhaps more tuition being given on interpretation of questions and the formal statements associated with the activities they were carrying out.

There was much more demand for the teacher to give basic tuition in number work, the understanding of decimal notation was a general cause for concern and the use of calculators became a teaching issue; only one of the students had his own



calculator. It would be true to say there were some students who had clearly decided that they were unable to do mathematics at all and thus were willing to take part in the computer based exercises but were reluctant to spend time on solving problems in their exercise books. However, the majority of the class were able to reach the stage of using the tangent of the angle to solve problems. Differences emerged only in the number of problems they were able to answer, and the great difficulty all of the group had with interpreting problems given in word form without diagrams. Their teacher, (Mr.B. was their usual class teacher) was surprised at the way they were able to use the computer effectively to correctly produce the table of sines and cosines and deduce that  $\sin A = \cos(90 - A)$  (though not in this form).

There was no gender issue in this group, with two boys and two girls being the most intuitive students and teacher time being equally divided between girls and boys. In the same way, the hands on experience on the computer was equally divided between boys and girls, except for one rather uninvolved girl who declined to use the computer at all.

### **7.7 Conclusions on the teaching of the experimental groups**

The teaching of four groups using the approach outlined above, across a supposedly wide ability range, prompted the following observations:

1. Apart from some ten or twelve persons in E1, who were remarkably acute in both their intuitive, holistic grasp of the concepts and their algebraic skills, and approximately three pupils from E4, who appeared unwilling to risk investing in an intuitive approach, the students across the ability range showed much more similarity in the way they were able to appreciate the visually based concepts than in the way they were able to use algebraic manipulation to solve problems.
2. Similarly, a difference emerged in the way the students were able to interpret word problems and construct appropriate diagrams. In E1, this was a minor problem, in E4 it was the major problem, far more of a concern than any other aspect of the teaching programme.
3. The extent to which the pupils considered mathematics to be a matter of learning procedures had a major effect on the time they were willing to spend on a gaining a more meaningful conceptual understanding. Though this was particularly evident with some of the boys in E2, it was a factor in the teaching of all four of the groups.



## 8. Post Test 1 Results

### 8.1 Preamble

Post test 1 was designed to assess the effects of teaching using the interrelating aspects of visual and numerical processing to produce a versatile learning environment. The null hypotheses being that: there would be no difference between the performances of the control and experimental groups in their understanding of trigonometry, and that there would be no difference in performance between the girls in the experimental groups and the girls in the control groups when considered separately, and the boys in the experimental groups and those in the control groups when considered separately, in their understanding of trigonometry as specified in the initial objectives. This demands a testing procedure which reflects these objectives, and is seen to reflect them by the teachers concerned with the teaching.

To this extent it was vital that the teachers accepted that the test would indeed reflect what they had intended the outcome of their teaching to be. The format of the test was discussed and agreed by the teachers of the control groups before it was written, and test papers were given to each teacher two days before they asked their classes to complete the paper. The control groups had been occupied with answering routine questions based on the efficient use of visually cued procedures for much of their time, and this was seen to reflect

the needs of the external examination. In order to properly reflect this emphasis the test was decided into two parts which can be described as follows:

The first section would be similar to much of the pre-test and would aim to discover the extent of the conceptual understanding of sine, cosine and tangent and the basic understanding of similarity of triangles upon which trigonometry is based.

The second section would consist of questions based on the routine exercises which had been completed in the class lessons, though much more time had been spent and much more practice given, on this aspect of the work in the control groups than in the experimental groups.

Breaking the test up in this way allowed the perceived strengths of the control group teaching, in the eyes of the teachers, to be compared separately. The teachers concerned were very happy with this arrangement. It also allowed the question of whether the teaching of the experimental groups had inhibited their ability to carry out these procedures to be analysed.

The problems associated with applying one test to a complete range of ability were discussed, particularly in terms of the likelihood of some pupils in the lower sets performing poorly and having their confidence dented. It was suggested that once more the full details of the research procedure should be given to the students, emphasising the fact that it was not a school test which would be recorded for school



purposes and that not every student would be expected to be able to answer all of the questions, though it was intended that the questions would begin at a relatively easy level. Nevertheless, there is a difficulty in separating research based testing of teaching and learning, and assessing student ability, at least in the eyes of the students. The benefits of being able to compare students from different ability groups, in terms of the effectiveness of the teaching, made a common test important, but it was a cause for concern to the teachers and the researcher.

Although the specific questions on the second part of the test were not given to the teachers until relatively late in

the teaching session they were assured that the questions selected would not be different in style from the ones their classes would have seen in their lessons. The questions for the first section were not discussed in detail, as they would lend themselves to specific coaching, but the initial objectives were reviewed and it was accepted that questions could safely be set on any of these objectives, with the proviso that students in C3 and C4 would have difficulty in understanding the way in which the length of the hypotenuse of the triangle can be found.

Mr. A the teacher of C1, and Mr. B the teacher of C2 both felt confident about the likely success of the majority of their students, though Mr. B expressed some doubts about the likely performance of four pupils who had expressed their concern about the difficulty of the topic. The teachers of C3 and C4

were concerned with the difficulty their pupil appeared to be having in "retaining the ideas", that is pupils who appeared, to be able to complete calculations one day became confused about similar questions given a day or so later.

The tests for E1 and C1 were given two or three days after the end of the two week teaching session, that is into the third week of the programme, but were not marked until the end of the following week, so the results were not available until the E2 and C2 class were about to reach the end of their two week session. In the same way the results for E2 and C2 were not available until E3 and C3. were reaching the end of their teaching sessions. This pattern continued until the final results were given two weeks after the final teaching sessions had been completed. Although the possibility could not be eliminated entirely, each teacher was asked not to discuss the particular questions with other teachers who had not yet given the test. For the experimental groups the tests were administered by the normal class teachers, as the researcher was occupied teaching the next class.

All of the tests were marked by the researcher then returned to the class teacher who checked the marking and returned them to the class, which in itself was seen to disadvantage the experimental groups, for the teachers of the control groups were free to discuss the test and the errors made by students in the group, thus preparing them in some way for the second post-test, whereas the experimental groups had only a brief visit by the researcher to talk about the results in general.



## 8.2 The test

The first section of the test, to be completed without the use of a calculator, was intended to assess the understanding of:

1. The fact that two triangles will have the same angles if the sides of one are exactly half the length of the corresponding sides of the other.
2. The sine of an angle, can not be greater than 1, though the tangent can
3. The sine of an acute angle is the same as the cosine of its complement.
4. The sine of 45 degrees is the same as the cosine of 45 degrees.
5. In a triangle the smallest side is opposite the smallest angle, in particular, the angle facing the smallest side in a right angled triangle will be less than 45 degrees.
6. When the hypotenuse of a right angled triangle is less than one unit, the sine of the angle can be calculated if the opposite side is known. Though this may appear to be nothing different from other calculations, the notion of scaling the triangle up until the hypotenuse became one unit would make the calculation relatively simple.
7. How deductions about lengths of sides, and trigonometric expressions can be made from diagrams without the use of calculators or tables.
8. The expression for an unknown hypotenuse can be written in terms of a known opposite side and the angle.

The second section consisted of three questions based on applying the information gained from diagrams to find missing sides or angles using the cosine and tangent of an angle. Each of these was similar to questions in the text book.

In the same way, the word problems, Q12 and Q13, test the skill of the student in interpreting the questions, forming appropriate diagrams and solving the resulting equations.

The question papers are given in full on the following pages.

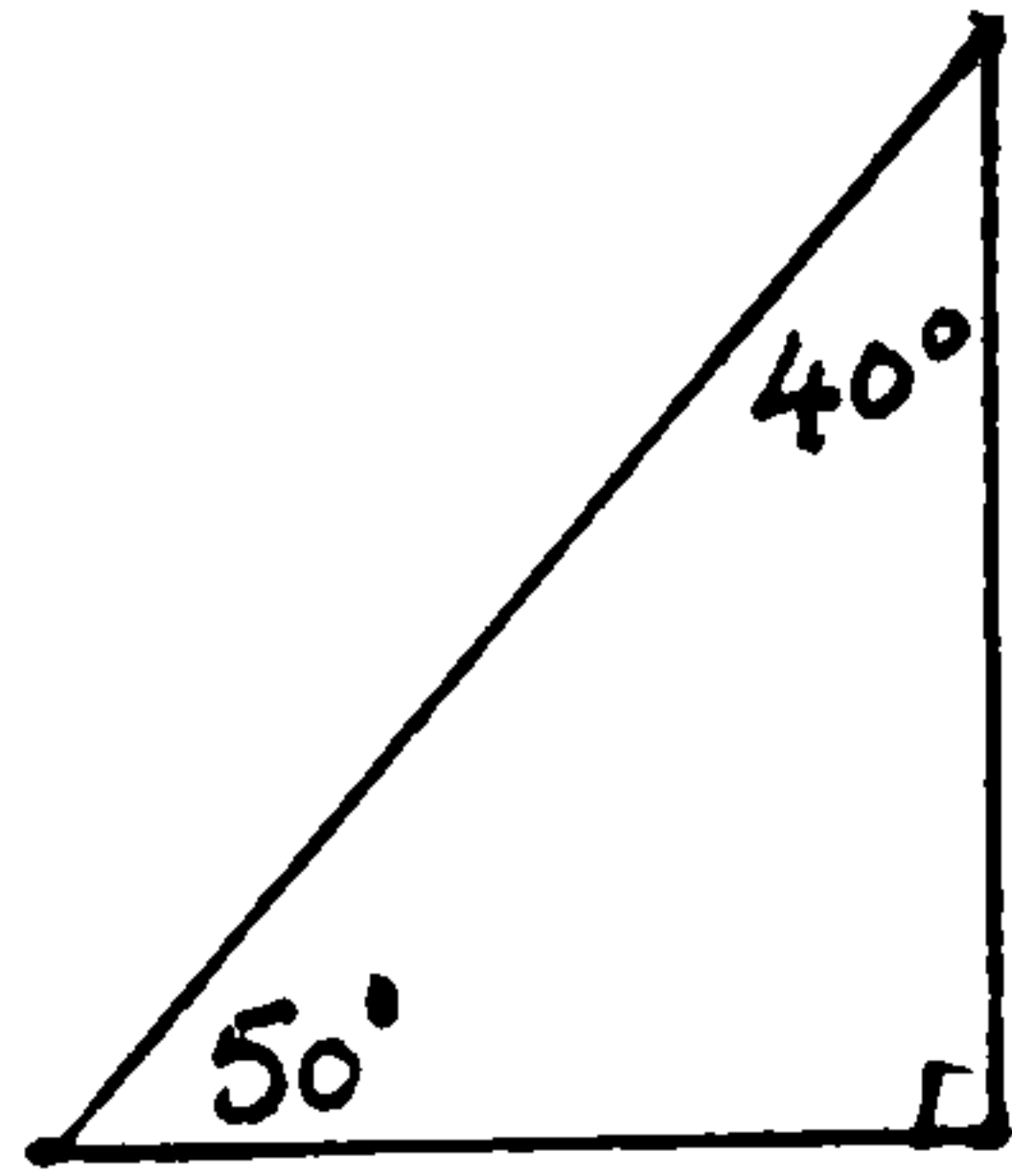


Post-Test 1.

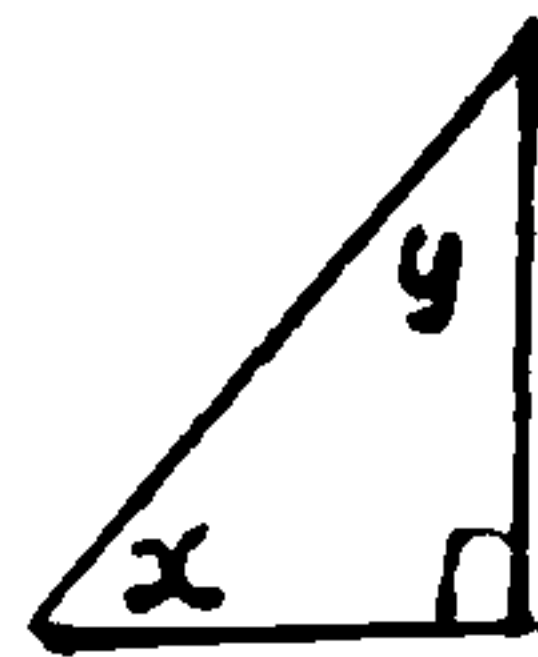
Section A

Approximately 20 mins for section A. You will not need a calculator for section A

Q1 Triangle 1



Triangle 2



Triangle 2 has sides which are exactly half the length of the sides of triangle 1. What are the missing sides in triangle 1?.

(Write angle  $x =$                   angle  $y =$                   )

Q2. Which of these statements can not be true?

- a)  $\sin x = 0$
- b)  $\tan x = 2$
- c)  $\cos x = 1$
- d)  $\sin x = 1.5$
- e)  $\cos x = 0.5$

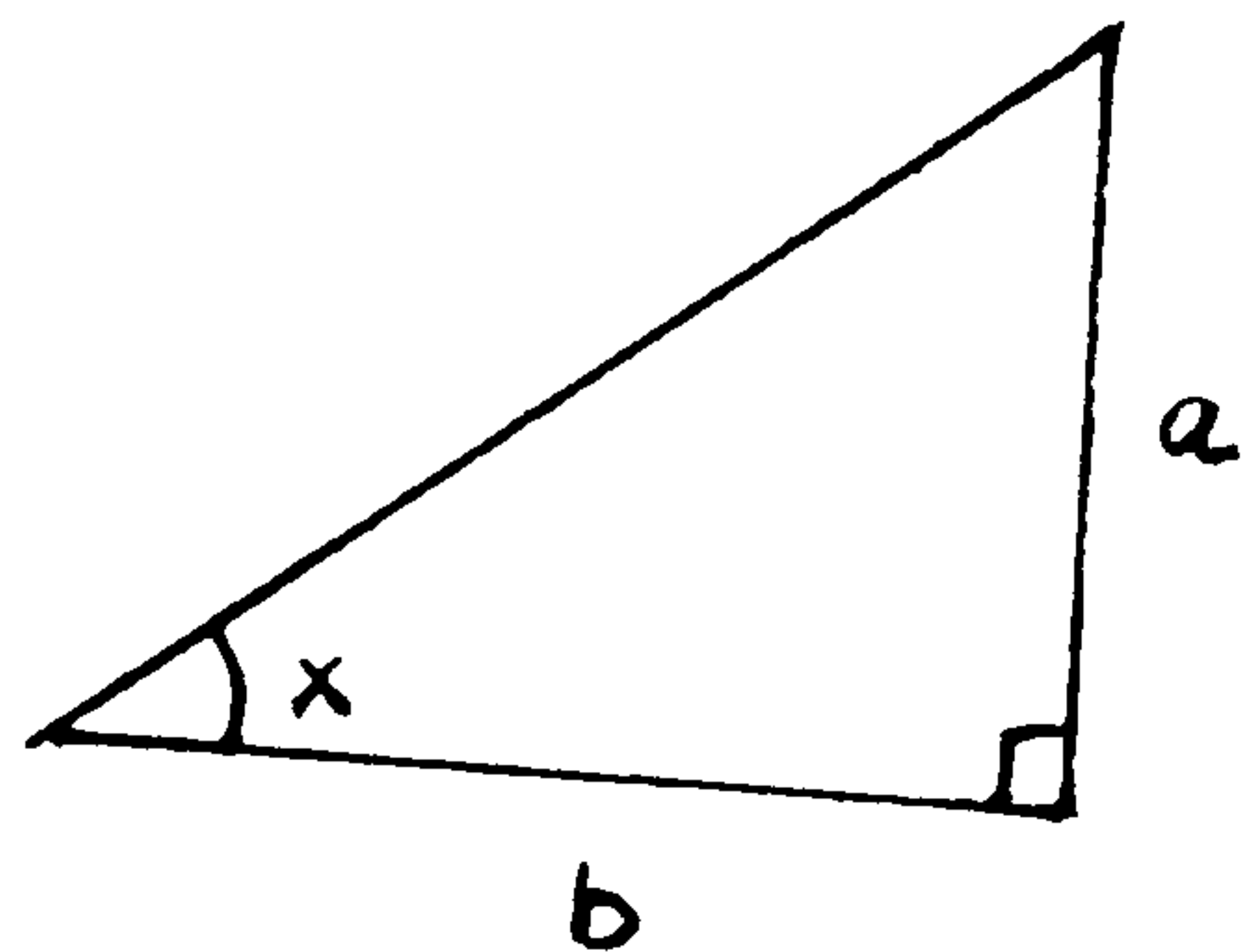
Q3 Complete this statement:

"The sine of 30 is the same as the cosine of....."

Q4 If the sine of  $x$  is the same as the cosine of  $x$  then  $x$  must be what value?

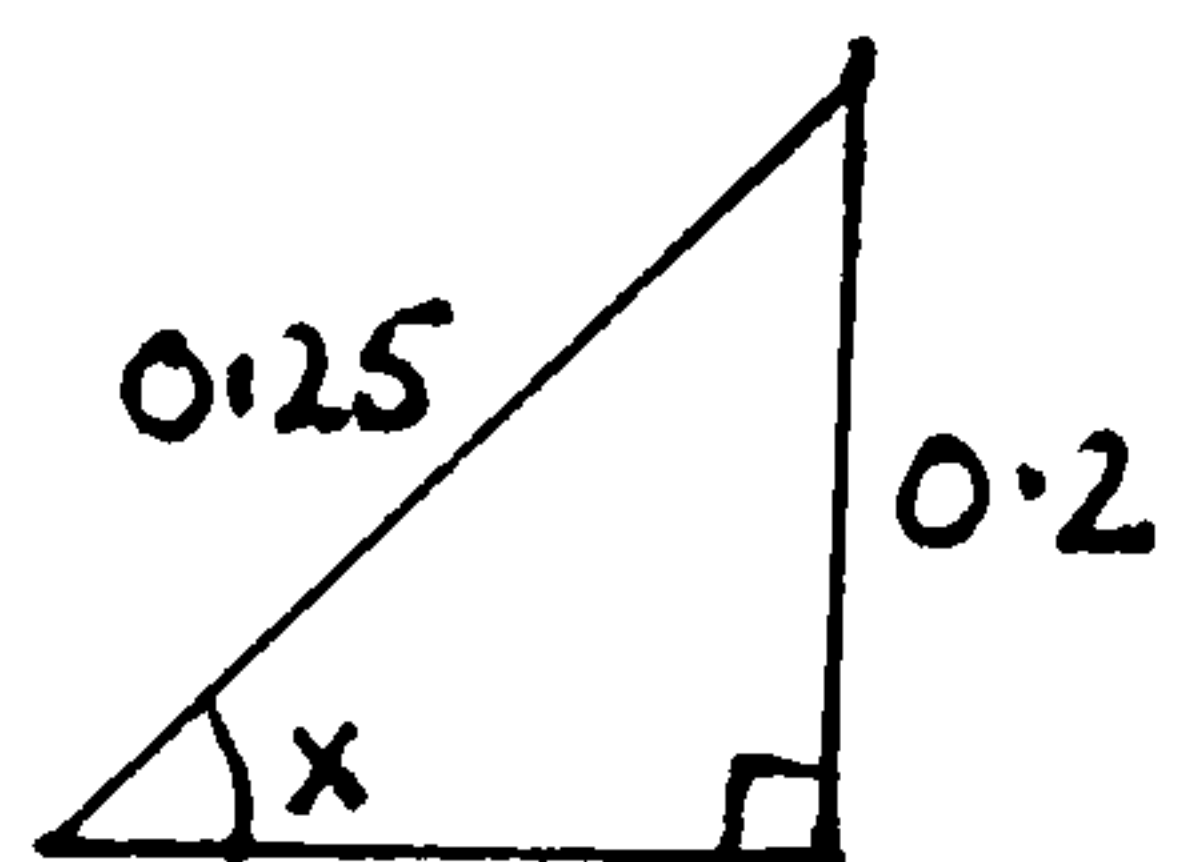
Q5.

If  $a$  is smaller than  $b$  what do you know about angle  $x$ ?

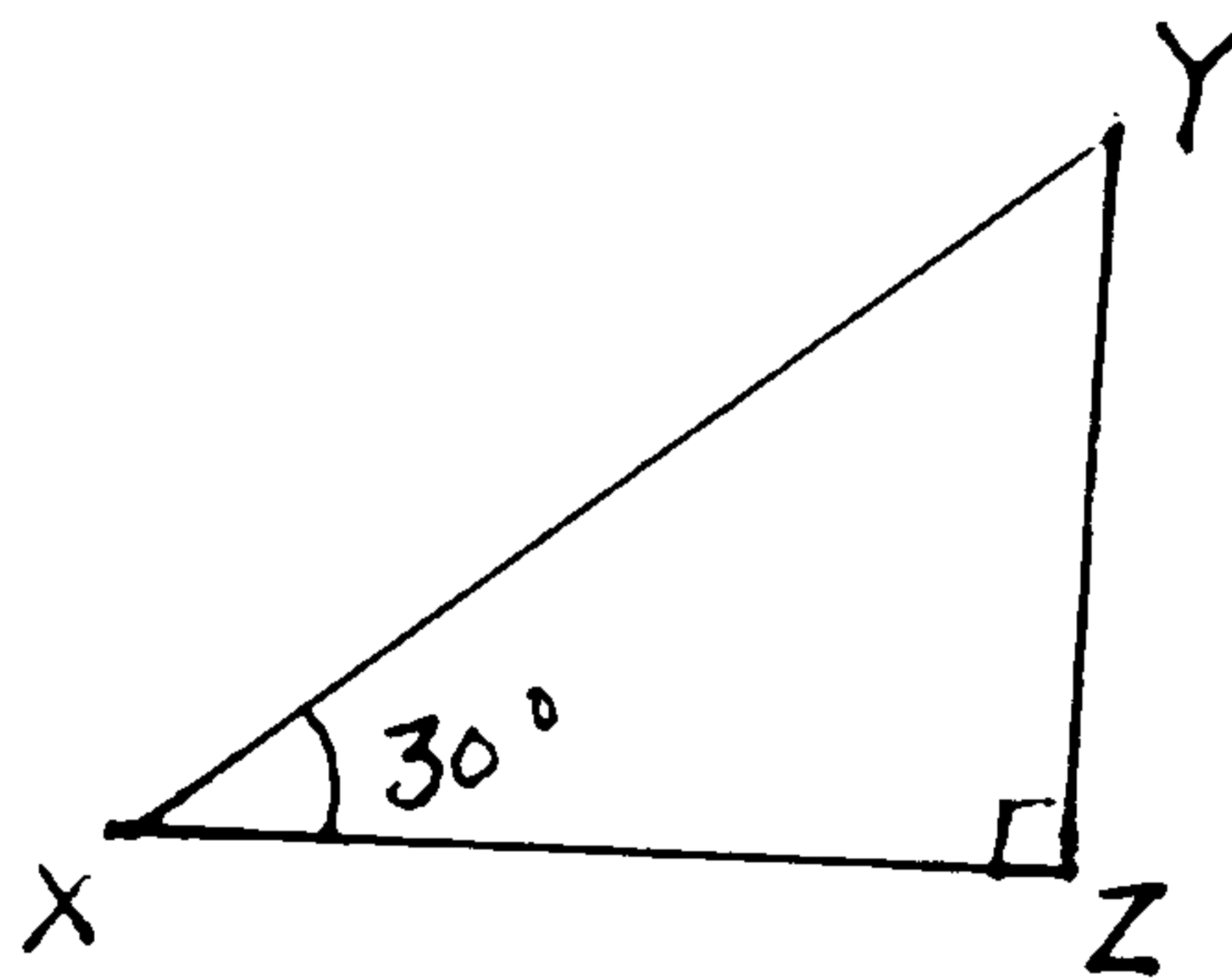
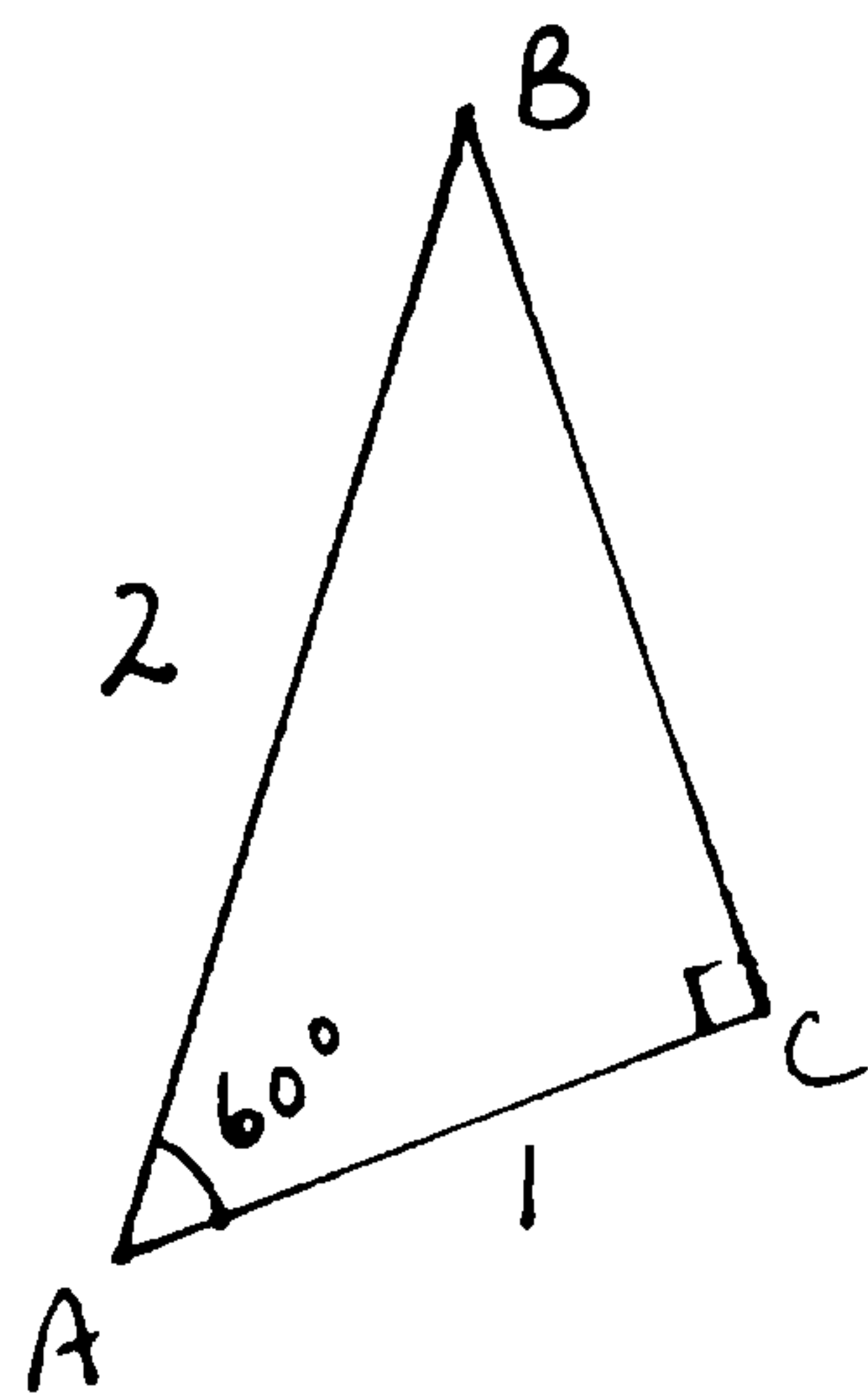


Q6

What is the sine of  $x$ ?



Q7.

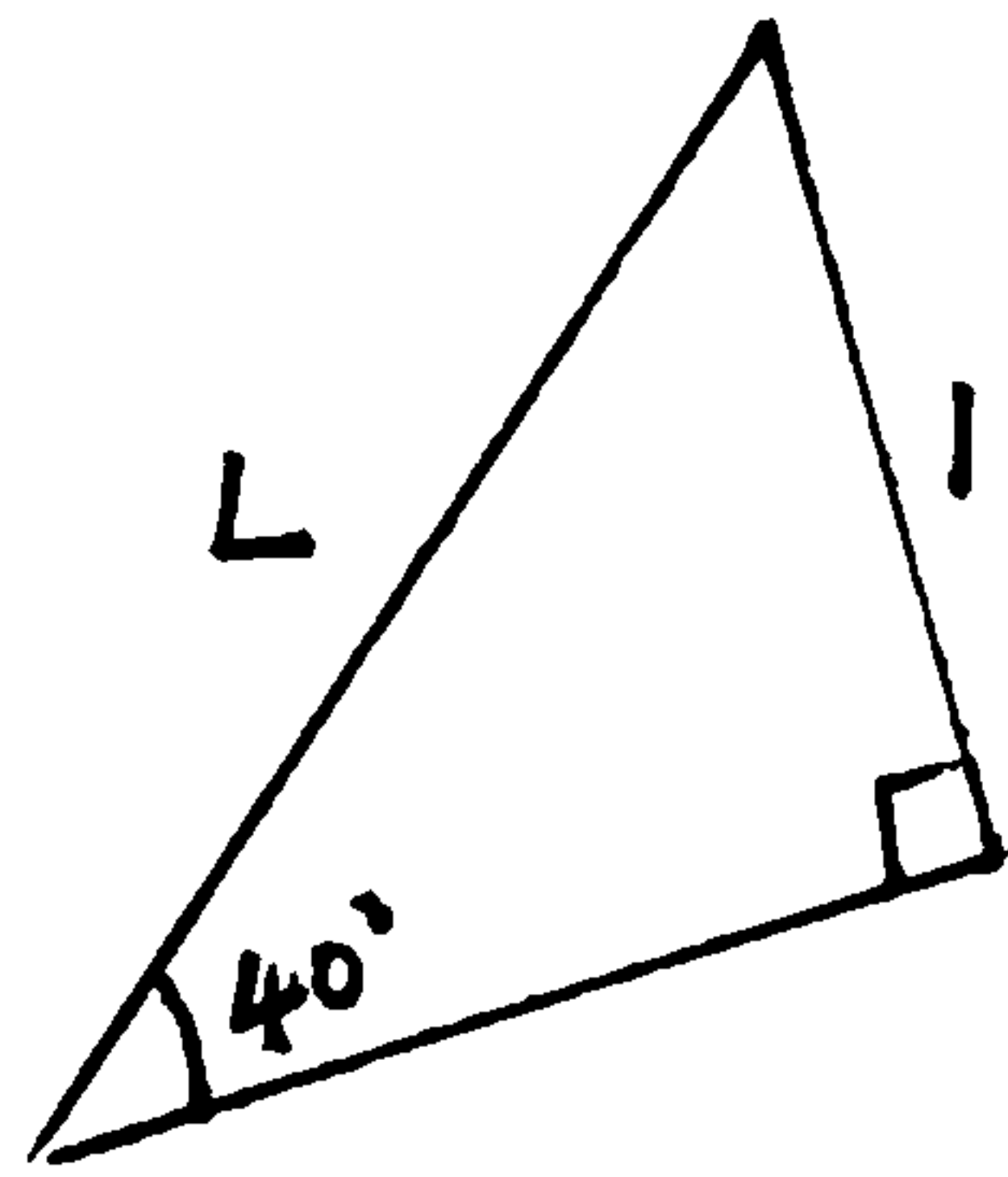


Write 'True' or 'False' for each of these statements:

- a)  $AC = \cos 60$
- b)  $XZ = 0.5$
- c)  $XZ = \sin 60$
- d)  $BC = 2\sin 60$
- e)  $YZ = \frac{1}{2} \sin 60$
- f)  $BC = \tan 60$



Q8.



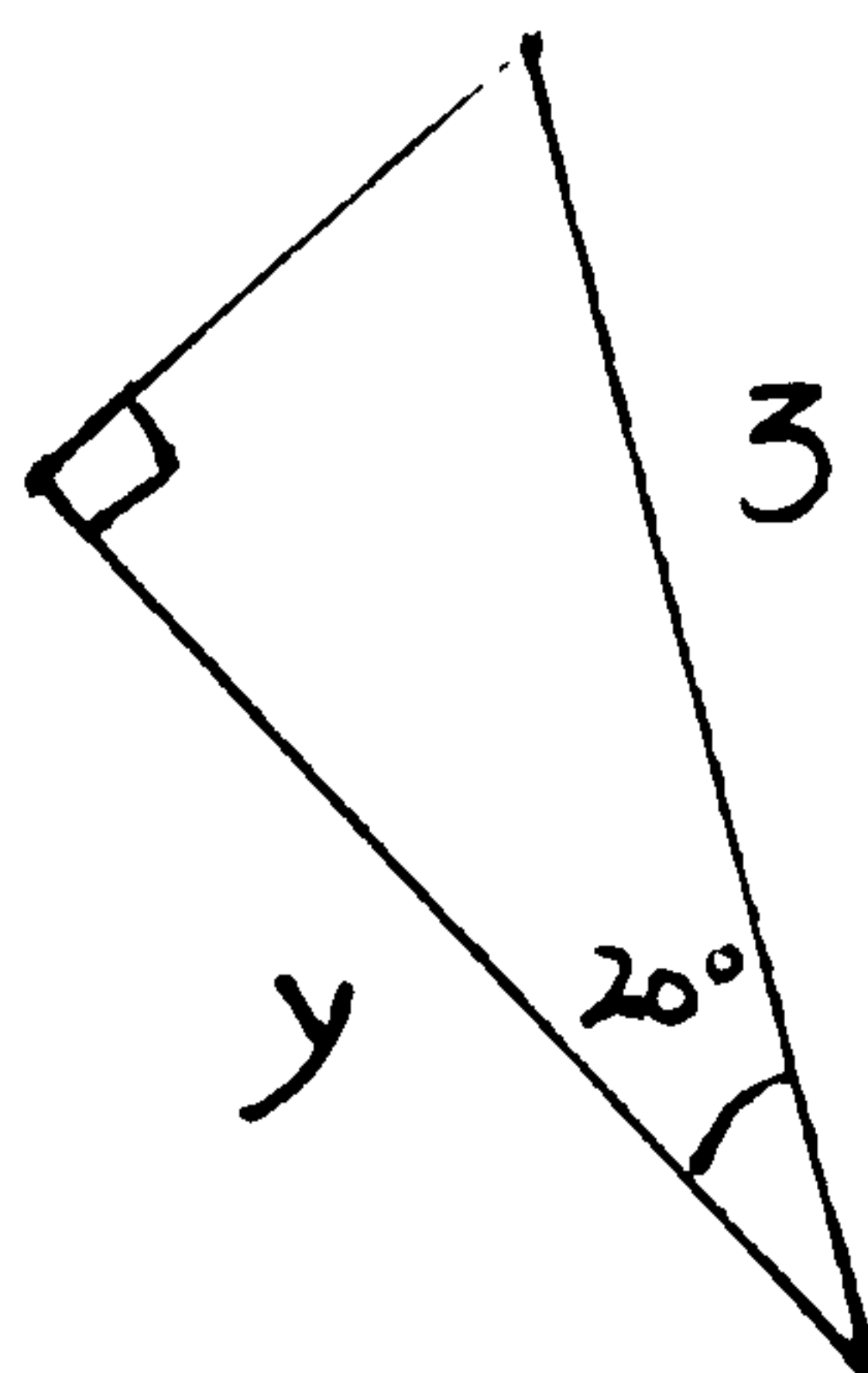
Which ONE of these statements  
is true?

- a)  $L = \cos 40$
- b)  $L = \sin 40$
- c)  $L = 1 / \sin 40$
- d)  $L = 1 / \cos 40$

### Section B

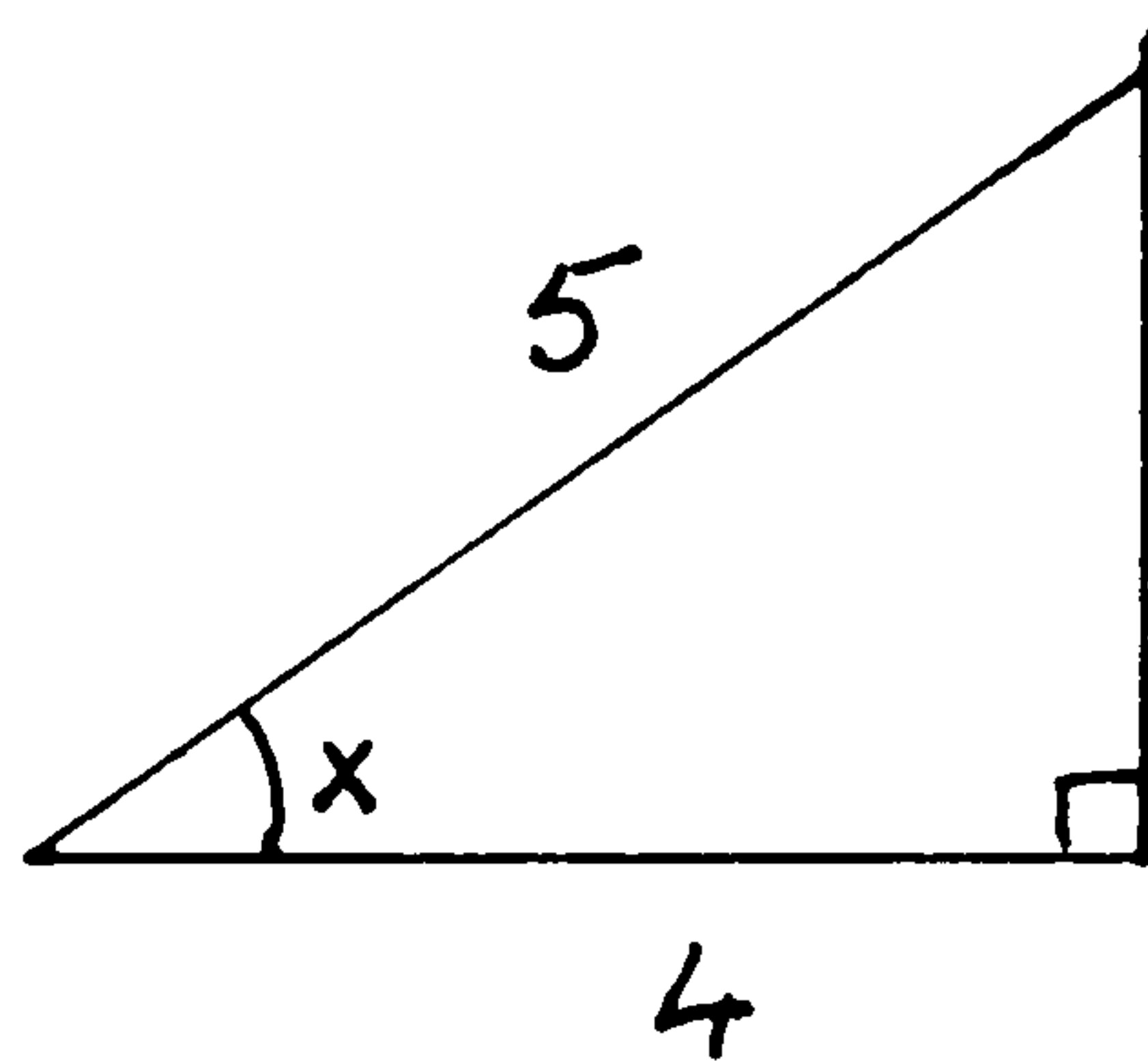
Use a calculator to help you to solve these problems.

Q9.



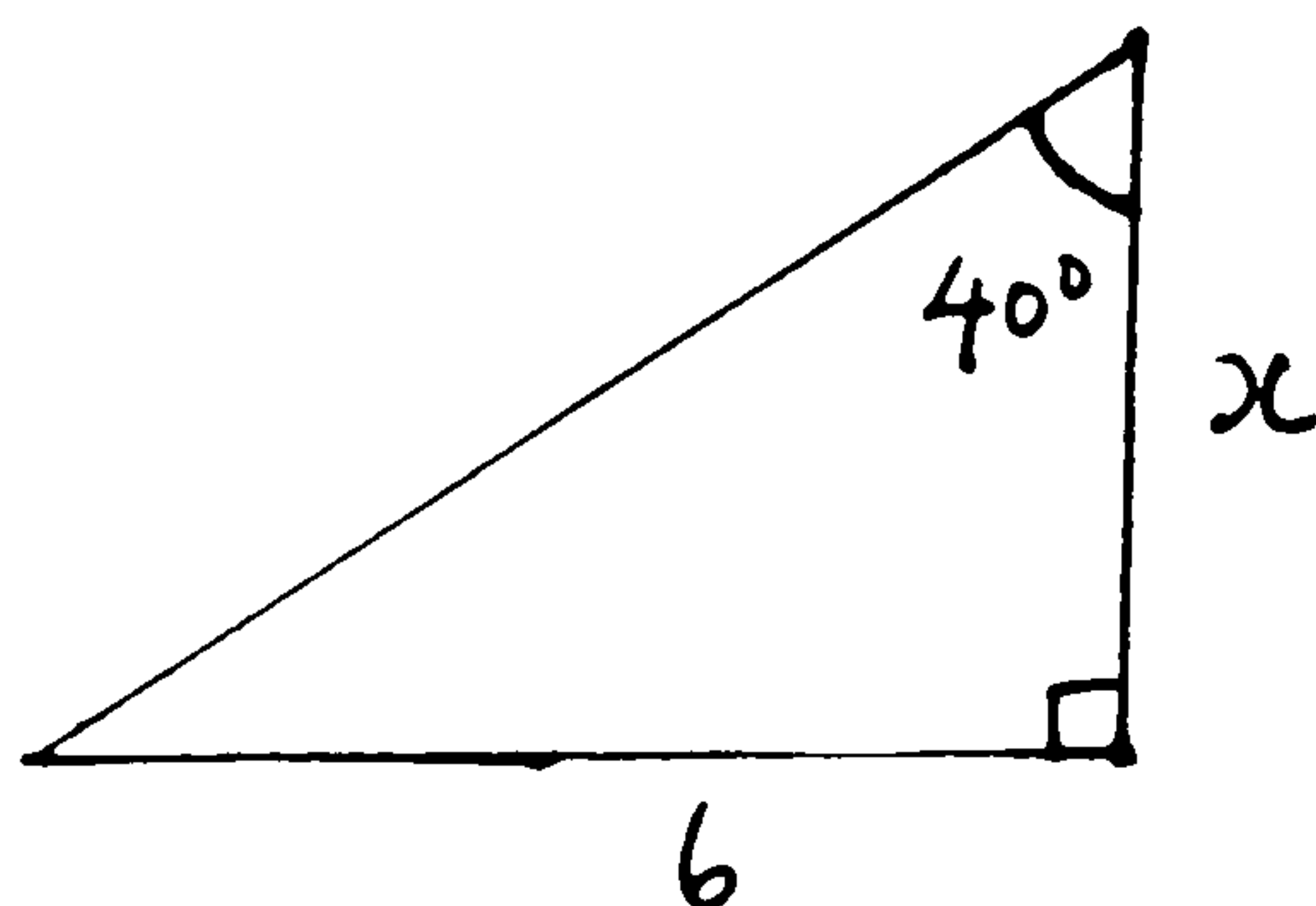
Find the length  $y$

Q10



Find the angle  $x$

Q11.



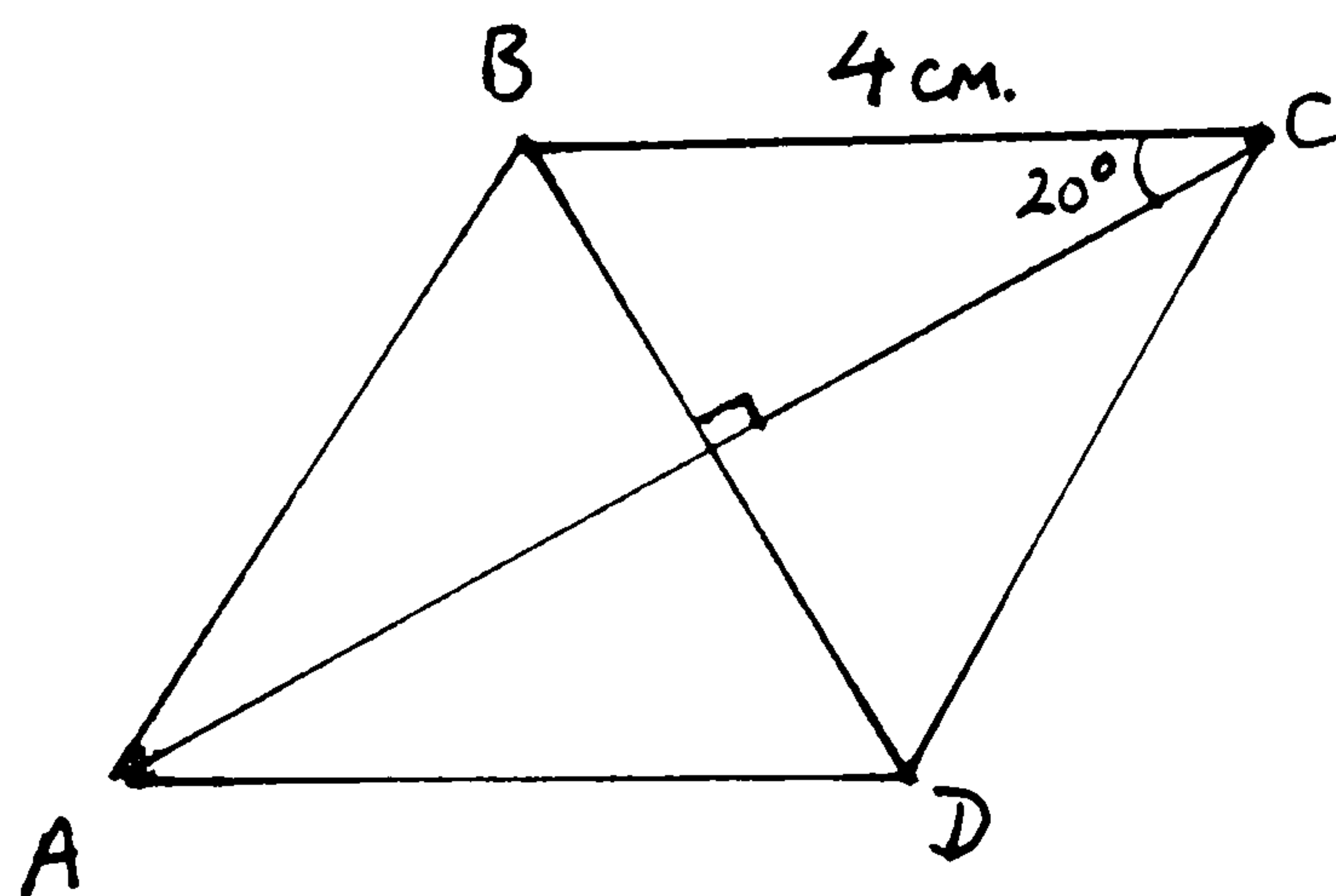
Find the side  $x$

Show your working for these three questions

Q12. A ship sails on a bearing of  $245^\circ$  for 30km. How far South has it sailed?

Q13 From the top of a cliff 30m high a man has to lower his telescope through 40 degrees to the horizontal to focus on a ship out to sea. How far is the ship from the foot of the cliff? (The cliff is vertical and you can ignore the height of the man)

Q14. The diagonals of this rhombus bisect each other at right angles. Each side of the rhombus is 4cm. How long is the diagonal AC?





### 8.3 The results for E1 and C1

Each question was marked and recorded on a grid for each class, thus the class scores for each question, as well as the individual pupil scores for each section could be examined. In addition the performances of boys and girls could be compared for individual sets and for the two samples, experimental and control

8.3.1 Comparison of responses to individual questions for E1(n=25, 16 girls, 9 boys) and C1(n=30, 17 girls, 13 boys)

Q1. a single mark given for correct angles.

E1 100% correct

C1 93% correct- two pupils halving all two angles though accepting that the 90 degree angle would be unchanged.

Q2 2 marks for selecting the statement which could not be true.

E1 68% correct- the largest mistaken suggestion was that  $\sin x = 0$  could not be correct 21%

C1 0.03% (one pupil)- again, the largest error, 61% was that  $\sin x = 0$  could not be true.

Q3 two marks

E1 100%

C1 60%- 27% suggesting that the sine of thirty degrees is the same as the cosine of thirty degrees.

Q4 two marks

E1 84%

C1 33%

Q5 two marks for suggesting that it is smaller than 45 degrees, one mark for suggesting that it is the smallest angle, or any other sensible interpretation.

E1 92% two marks 4% one mark

C1 63% two marks 10% one mark

Q6 two marks for calculation of 0.8, one mark for 0.2/0.25

E1 52% two marks, 8% one mark

C2 17% two marks, 47% one mark.

This question suggests the extent to which the experimental group were able to move away from the trigonometrical ratios associated with the mnemonic they had learnt, back to the fundamental scaling up procedure to produce the hypotenuse of one unit. Of the pupils gaining two marks from the experimental group, not one had written 0.2/0.25 and proceeded to calculate the answer as 0.8, all had written the answer directly. In the control groups all of those who gained marks wrote 0.2/0.25 but most were unable to proceed to the final answer.

Q7 0.5 marks for each correct answer, giving a maximum mark of three. Inevitably there would be a certain amount of guessing with this form of question, distorting an estimate of the amount of understanding being displayed. The mean marks of the



two groups and the numbers scoring 2.5 or 3 marks give some indication of the difference between the groups.

E1 mean mark 1.92, 40% scoring 2.5 or 3

C1 mean mark 1.48 (below the level associated with the expectation when selecting by chance), 17% scoring 2.5 or 3 marks.

Q8 two marks for selecting the correct answer.

E1 79% correct

C2 47% correct.

This question indicates how difficult the pupils in the control group found it to manipulate the equation  $\sin 40 = 1/L$  into the form where L is the subject, despite some time being spent on exercises based on this kind of manipulation.

The first eight questions were marked out of a total of sixteen and the results compared. Using the analysis of variance described earlier. The results are given in detail at the end of this section, though it can be summarised in terms of the existence of a significant difference between the two groups, ( $p < 0.05$ ), with the experimental group having the superior performance.

It can be seen that the attempts by Mr.A to present the topic in such a way as to allow the class to reach a conceptual understanding of trigonometry by reflecting on the routines which had been developed in using the trigonometrical ratios

had not been successful when compared with the combined visual and algebraic approach used with the experimental group.

The second section, provided the routine questions which had been considered the strength of the teaching of the control groups. The results for these questions were:

Q9 Two marks for the correct solution, one mark for the initial correct statement without the correct working, or for the incorrect ratio which had been correctly calculated.

E1 88% two marks, 8% one mark, 4% no marks.

C1 90% two marks, 10% no marks.

Q10. marked in a similar way to Q9.

E1 76% two marks, 12% one mark, 12% no marks.

C1 83% two marks, 17% one mark, 0% no marks.

Q11. Marked out of three to allow for the extra line of algebraic manipulation necessary to find  $x$  from  $\tan 40 = 6/x$

E1. 40% three marks 52% one mark. 8% no marks. All of the pupils in this group were either able to identify  $\tan 50 = x/6$  or they mistakenly completed the calculation using sine or cosine of 40

C1. 66% three marks, 7% two marks, 17% one mark, 10% no marks.

Q12. Marked out of three. One mark given for a diagram or some indication of the acute angle used in the appropriate right angled triangle and the other two marks for identification of the trigonometrical ratio leading to a correct calculation.

E1. 88% three marks, 8% one mark, 4% no marks.

C1. 80% three marks, 6% two marks, 7% one mark, 7% no marks.

Q13 Marked out of three in a similar way to question 12.

E1 56% three marks, 4% two marks, 32% one mark. 8% no marks.

C1 27% three marks, 3% two marks, 63% one mark, 7% no marks.

Q14 Marked out of three

E1 96% three marks, 4% two marks.

C1 87% three marks, 7% two marks, 6% one mark.

This second section was marked out of sixteen and the mean marks compared using the same analysis of variance test. The mean marks were extremely close for this section. Experimental group 80.5%, control group 80.2%. Clearly there is no significant difference in the mean scores for the two groups in this section. However, before examining the degree of mastery shown by the most able pupils in these groups it may be advantageous to examine the comparative scores for the above section of the test more closely and attempt to draw some tentative conclusions.

The more simple routine questions demanding a knowledge of the trigonometric ratios and the identification of the three sides of the triangle, in terms of the opposite, adjacent and hypotenuse, that is questions 9,10 and 11, were indeed well answered by the members of the control group. A higher percentage of the control group than the experimental group



obtained the correct answer to all three of these questions. In questions 10 and 11 in particular more of the experimental group used the incorrect trigonometrical ratio than those in the control group. One may conclude that the members of the control group had memorised the mnemonic well and could apply it efficiently to this form of question. However, when word problems were given, or a slightly more involved diagram, as in questions 12,13 and 14, more of the experimental groups were able to complete the calculations correctly. It may appear then that practice in this form of problem is not the way to ensure that similar problems can be completed at a later date, for the experimental groups had in fact spent a much smaller proportion of their teaching time attempting questions such as this.

Q11 highlighted the effect of the perceived lack of time spent on discussing the tangent of the angle in the teaching of E1. The ten pupils who completed the question correctly, neatly used the tangent of the complementary angle and solved the question in one line; the others were unsure about the tangent of the angle and used the cosine of 40 degrees. The difference in approach was marked, for none of the control group used the complementary angle in order to find  $x$ .

To examine the degree of mastery of the subject amongst the best performers in both groups it may be beneficial to compare the numbers who scored the higher marks in the two sections and the total.

Section A	11-12	13-14	15-16
E1	28%	8%	36%
C1	20%	3%	0%

Section B	11-12	13-14	15-16
E1	24%	16%	40%
C1	37%	33%	20%

Total	24-26	27-29	30-32
E1	12%	12%	28%
C1	23%	7%	0%

It can readily be seen that the experimental group provided a greater proportion of the class who could be seen as having 'mastered' the topic, not only in section A but in section B, which was considered to be the strength of the teaching of the control group. However, the teaching of the control group was relatively successful in giving some students the procedures they needed to answer the more straightforward questions in section B, so that the proportion of each class gaining more than 10 marks in this section favoured the control group. (E1 81%, C1 90%).

### 8.3.2 Statistical analysis E1 and C1

Analysis of these results in detail provides the following information.

Using two tailed analysis of variance, with null hypothesis that there will be no difference between the performance of the two groups. level of significance  $p < 0.05$ .

Two sample test, between sample degrees of freedom 1, within samples degrees of freedom 53. (Degrees of Freedom abbreviated to d.f. henceforth)

In the following tables, and throughout the analysis, V.E. refers to 'variance estimate'.

For section A: E1 mean 12.52, standard deviation 2.729

C1 mean 7.5667, standard deviation 3.0761

	d.f	sum of squares.	V.E.
between	1	334.575	334.575
within	53	470.107	8.870
total	54	804.682	

F.ratio 37.72.

The null hypothesis is rejected.

For section B: E1 mean 12.88, standard deviation 3.229

C1 mean 12.83, standard deviation 2.134

no significant difference in the results.

For complete totals: E1 mean 25.42 s.d. 5.01

C1 mean 20.4 s.d 4.53.



	d.f	sum of squares	V.E.
between	1	304.226	304.226
within	53	1272.502	24.009
Total	54	1576.727	

F.ratio 12.671

The null hypothesis is rejected.

#### 8.4 The results for E2 and C2

8.4.1 Comparison of responses to individual questions for E2 (n=26, 14 girls, 12 boys) and C2 (n=29, 12 girls, 17 boys)

Q1.A single mark given for correct angle

E1 96% correct

C1 90% correct

Q2 Two marks for selecting the correct statement

E1 35% correct

C1 21% correct

For both groups, as for E1 and C1 the largest incorrect suggestion was that  $\sin x = 0$  could not be true.

Q3 Two marks

E1 77% correct

C1 55% correct

Q4 Two marks

E1 77% correct

C1 35% correct- There was much wider variety of response from this group, 5 pupils suggesting 0 degrees, three suggesting 90 degrees.

Q5 Two marks.

E1 89% correct

C1 38% correct

Q6 Two marks for correct calculation, one mark for  $0.2/0.8$

E1 0% correct 4% one mark.

C1 7% correct, 28% one mark.

The failure of the experimental group to deal successfully with this question reveals the difficulty they had in using numbers less than one unit in the scaling process. To this extent the routine established by the control groups was sufficient for eight of the group to be able to write down an expression for  $\sin x$ , though they were unable to complete the calculation, and two pupils to write down the expression and go on to the correct answer. Clearly, the use of numbers less than one unit was enough of a difference from the questions practised by the control group to cause a crisis of confidence in the well-rehearsed routines for the 66% of the group to be unable to write down the initial expression. The members of the experimental group did not attempt to write down an expression

for  $\sin x$ , but were clearly looking for a scaling process which they were unable to complete.

Q7.E1 Mean mark 1.537, 8% scoring 2.5 or 3

C1 Mean mark 1.293 0% scoring 2.5 or 3

Q8 E1 77% correct

C1 28% correct

The analysis of the marks for this first section showed a significant difference in favour of the experimental group, with the null hypothesis being rejected ( $p < 0.05$ ). The details are given in the detailed analysis below.

The results for the second section were as follows:

Q9 E2 73% two marks, 8% one mark

C2 62% two marks, 10% one mark

Q10 E2 65% two marks, 12% one mark

C2 48% two marks, 17% one mark

Q11 E2 0% three marks, 4% two marks, 16% 1 mark

C2 3% three marks, 0% two marks, 48% 1 mark

Q12 E2 32% three marks, 8% two marks, 24% one mark.

C2 59% three marks, 10% two marks.



Q13 E2 0% three marks, 8% two marks, 56% one mark.

C2 17% three marks, 7% two marks, 0% one mark.

Q14 E2 64% three marks, 4% two marks, 12% one mark.

C2 48% three marks, 7% two marks, 17% one mark.

The differences in scores for this section reveal several points. In the more routine questions using the sine and cosine ratios(Q9 and 10), the experimental group actually performed rather better than the control group. However, the necessity for the recognition of the tangent of the angle, as in Q11 and 13 gave very real problems to both groups, but in particular the experimental group. It had been recognised that the introduction of the tangent of the angle using the visual approach was proving to be difficult in the time available and the response to these questions highlighted the insecurity of the members of this group when faced with using the tangent. Interestingly, the control group were adept at spotting that the tangent ratio should be used in question 11 but were unable, in general, to proceed beyond the initial statement  $\tan 40 = 6/x$ .

The control group had been well practised in problems similar to Q12 and were able to score more marks for this question than the experimental group.

#### 8.4.2 Statistical analysis of results E2 and C2

For section A: E2 mean 8.98                      s.d 3.018

C2 mean 6.03                      s.d 3.496

Using two tailed analysis of variance for the two samples.

Between sample d.f. 1. within sample d.f. 53

	d.f.	Sum of Squares	V.E.
between	1	119.003	119.003
within	53	591.206	11.155
total	54	710.209	

F.ratio 10.668

The null hypothesis, that there will be no difference between the performances of the two groups, is rejected (p 0.05)

For section B: E2 mean 7.269                      s.d 3.352

C2 mean 7.379                      s.d 4.046

The null hypothesis is accepted.

For the complete totals: E2 mean 16.25      s.d 5.37

C2 mean 13.41      s.d. 6.937

	d.f	Sum of Squares	V.E.
between	1	110.277	110.277
within	53	2146.659	40.503
total	54	2256.936	

F. ratio 2.723

The F ratio is not significant and the null hypothesis is accepted.

As with E1 and C1 there is a significant difference in favour of the experimental groups in section A of the test and little or no difference in the total marks gained for section B, though there are different areas of weakness in the more routine areas of calculation. The teaching of the experimental group E2 was thought by the researcher to be the least successful of the teaching sessions, for reasons described in the previous chapter, yet the performance of the experimental group in the section well rehearsed by the control group was very similar to that of the control group.

### 8.5 The results for E3 and C3

8.5.1. Comparison of responses to individual questions for E3(n=21, 13 girls, 8 boys) and C3(n=21, 10 girls, 11 boys)

Q1.E3 90% correct.

C3 57% correct.

The score for C3 on this question illustrates a remarkable inability for almost half of the group to recognise that corresponding angles are equal in similar triangles. The incorrect answers were invariably half of the angles in the first triangle.



Q2 E3 38% correct

C3 14% correct.

Q3 E3 95% correct

C3 52% correct

Q4 E3 71% correct

C3 10% correct

Q5 E3 57% two marks, 29% one mark

C3 14% two marks, 10% one mark

Q6 E3 0% two marks, 19% one mark.

C3 0% two marks, 19% one mark.

The interesting observation on this question is that the members of E3, who were unable to scale the triangle up so that the hypotenuse was one unit, reverted to the ratio notation and scored one mark. The members of C3, who had been taught to use the ratio notation were confused by the use of numbers less than one unit to the extent that only four members of the group, the same as for E3, were able to gain one mark.

Q7 E3 mean mark 1.25 ( four pupils missing out the question altogether thus scoring zero) 5% scoring 2.5, 29% scoring 2

C3 all pupils made an attempt at the question so no zero marks were scored, Mean mark 1.225. 0% scoring 2.5, 14% scoring 2.

Q8. E3 29% correct

C3 29% correct

These eight questions were marked out of sixteen and the results compared using the two tailed analysis of variance. The mean mark for E3 being 8.38 and for C3 4.43. There was a significant difference in favour of E3 ( $p < 0.05$ ) and the null hypothesis rejected. Then details are given below.

The marks for the second section were as follows:

Q9. E3 66.7% two marks, 5% one mark.

C3 57.1% two marks, 14.2% one mark.

Q10 E3 62% two marks, 9.5% one mark.

C3 42.8% two marks, 0% one mark.

Q11 E3 14.3% three marks, 19% two marks, 23.8% one mark.

C3 28.5% three marks, 10% two marks, 4.8% one mark.

Again the experimental group floundered on the use of the tangent of the angle, though 33.3% recognised that the tangent should be used, 19% failed to recognise that the complement of the angle was required.

Q12 E3 42.9% three marks, 19% two marks, 9.5% one marks.

C3 9.5% three marks, 0% two marks, 9.5% one mark.

There was a remarkable difference in the performance of the two groups for this question, not merely in terms of the marks

awarded but in terms of the diagrams drawn to represent the solution. The members of C3 had clearly practised bearings questions before but many were unable to select the triangle to be used and many, (some 28.5% of the group), tried to use the bearing 245 degrees as if it were an acute angle in a triangle.

Q13 E3 14.3% three marks, 4.8% two marks, 28.6% one mark.

C3 9.5% three marks, 14.3% two marks, 9.5% one mark.

Q14 E3 19% three marks, 23.8% two marks, 14.3% one mark.

C3 19% three marks, 4.8% two marks, 0% one mark.

This question showed a clear difference in favour of the experimental group in recognising the sine of the angle should be used in carrying out the necessary calculation. Those who received only two marks were those who failed to recognise that  $4\sin 20$  should be doubled to give the length of the diagonal.

The difference in performance between these two groups was marked, not only in the more general conceptual questions but in the more routine questions favoured by the teachers of the control groups. The results are given in detail below:

#### 8.5.2. Statistical analysis of results.

Using two sample analysis of variance, two tailed, with level of significance  $p < 0.05$ . Between groups d.f. 1. within groups



d.f. 40. The null hypothesis being that there is no difference between the performance of the two groups,

For section A: E3 mean 8.38      s.d. 2.5256.

C3 mean 4.43      s.d. 2.3365

	d.f.	sum of squares.	V.E.
between	1	164.024	164.024.
within	40	248.595	6.215
total	41	412.619	

F ratio 26.3921

The null hypothesis is rejected.

For section B: E3 mean 7.5238      s.d 3.8498

C3 mean 4.9524      s.d 4.0997

	d.f.	sum of squares	V.E.
between	1	69.429	69.429.
within	40	664.19	16.605
total	41	733.619	

F ratio 4.181

The null hypothesis is rejected.

For totals: E3 mean 15.9      s.d. 5.42

C3 mean 9.4      s.d 5.76

d.f	sum of squares	V.E.
between 1	427.524	427.524
within 40	1315.595	32.89
total 41	1743.119	

F ratio 12.998

The null hypothesis is rejected.

The important difference between the results of these two groups and the results of the other two pairs of groups is that the significantly better performance in the first of the sections is repeated in the more routine questions assumed to be the strength of the teaching associated with the control groups. It appears that for this level of ability the groups taught by the more algebraic process, leading to the practice of routines, not only failed to be able to reflect on the concepts in order to obtain a more conceptual understanding of the work but failed to apply the routines to questions even marginally different from the ones they had practised.

## 8.6 The results for E4 and C4

### 8.6.1. Comparison of responses to individual questions for E4(n=12, 6 boys, 6 girls) and C4(n=14, 5 boys, 9 girls)

Although the teachers giving the tests were instructed to help the pupils who had difficulty reading the questions this was evidently still a problem for some pupils in both of these groups. One girl from E4 wrote that she "could not understand

the questions" and left the rest of the page blank. However, there was a very marked difference in the kind of answers proffered by the pupils in the control group to those from the experimental group in that very many of their answers were not merely incorrect but apparently nonsensical, giving angles in terms of centimetres or obtuse angles in a right angled triangle.

Q1. E4 83% correct

C4 29% correct.

This is perhaps the clearest indication of the misunderstanding developed by the members of C4. Despite a teaching approach based on drawing similar triangles, the great majority of the class assumed that the angles must be halved if the sides are halved. Only one member of E4 made this deduction.

Q2. E4 58% correct

C4 7% correct.

Q3 E4 75% correct

C4 64% correct

The high number of correct responses to this question, from the control group marks some degree of success in the approach adopted by the teacher in drawing conclusions from the tables made up from the pupils' drawings, though the experimental group were more successful.



Q4 E4 75% correct

C4 0% correct.

It is difficult to see why the control group having the relative success in Q3 should provide no correct answers at all to this question.

Q5. E4 42% two marks, 25% one mark.

C4 0% two marks, 7.% one mark

Q6 E4 8% two marks, 17% one mark

C4 14% two marks, 0% one mark.

Two pupils from C4 were able to work through from  $\sin x = 0.2/0.25$ , unperturbed by the numbers involved being less than one unit.

Q7. E4 mean mark 1.58, 16.7% scoring 2.5 marks, 50% scoring 2 marks.

C4 mean mark 0.68, 0% scoring 2.5 marks, 7% scoring 2 marks.

Q8 E4 0% correct

C4 0% correct.

Both groups were completely confused by the hypotenuse being unknown.

The results for section A revealed that the experimental group had a mean score of 7.75, whilst the control group had a mean

of 2.67. This represents a markedly significant difference, with the experimental group performing as well as C1 on this section, despite the fact that one member of the group had not attempted to answer any questions.

The second section was predictably problematic for both groups and the percentage of correct solutions for each question was very low. However, an important point which should be emphasised is that the correct answers given in E4 were consistently given by the same four pupils, which gave some high total scores for some pupils who had previously been assumed to be unable to understand work of this nature. Further discussion on this important point is given in the conclusion to this chapter. The marks for section B were as follows:

Q9. E4 25% two marks, 17% one mark

C4 17% two marks, 0% one mark

Q10 E4 17% two marks, 8% one mark.

C4 17% two marks, 0% one mark

Q11 E4 8% three marks, 17% one mark

C4 no marks scored for this question.

Q12 E4 17% three marks, 17% one mark

C4 no marks scored

Q13 E4 8% three marks, 17% two marks, 8% one mark.

C4 no marks scored for this question

Q14 E4 8.% three marks, 17% two marks, 17% one mark.

C4 no marks scored for this question.

#### 8.6.2. Statistical analysis of the results for E4 and C4

Using two sample two tailed analysis of variance, between groups d.f. 1 within groups d.f. 24 , level of significance ( $p < 0.05$ ). The null hypothesis is that there is no difference in then performances of the two groups.

For section A: E4 mean 7.75            s.d. 3.04

C4 mean 2.67            s.d. 1.76

	d.f.	sum of squares.	V.E.
between	1	217.012	217.012
within	24	152.655	6.106
total	25	369.667	

F ratio 35.54. This shows significantly difference ( $p < 0.05$ ) and the null hypothesis is rejected.

For section B: E4 mean 3.58            s.d. 3.964.

C4 mean 0.714            s.d. 1.111

	d.f.	sum of squares.	V.E
between	1	70.304	70.304
within	24	192.659	7.706
total	25	262.963	



F ratio 9.123. This shows a significant difference,  $p < 0.05$ , and the null hypothesis is rejected.

For totals: E4 mean 11.33                      s.d. 6.577  
                    C4 mean 3.384                      s.d. 2.0314.

	d.f.	sum of squares	V.E.
between	1.	430. 230	430.230
within	24	584. 067	23.363
total	25	1014.296	

F ratio 18.415

The null hypothesis is rejected.

Of all the comparisons made between the experimental and control groups this was the only one where there was a significant difference ( $p < 0.05$ ) between the standard deviations of the two groups as well as significant differences on the means of each section. The implication here being that the two groups do not appear to be taken from the same population. The results were so remarkably different at this level of ability that it would be difficult to assume that they were indeed from the same background population. However, the class tests used to place pupils in these sets were consistent and applied throughout the two halls, as described in chapter 6.

The analysis of variance is a robust test when the numbers in each sample are approximately equal, even though the standard deviations differ significantly, so the above analysis is valid.

Though there were obvious difficulties for many members of the experimental group in solving the problems in section B of the test, the answers to section A indicated a conceptual understanding of the basis of trigonometry amongst many members of the group. The failure to answer questions in section B, despite success in the first section leads to the conclusion that in many cases it is the form in which the question is given which has a major bearing on the success rate at this level of ability.

Particular individuals in groups for which the topic had been considered too difficult had made encouraging improvement in previous research using computer graphics, (Blackett N, 1987) that is they appear to gain an understanding of concepts which had been assumed to be too difficult for them and proceed to perform far better than teachers would have anticipated. In this case four pupils( two boys and two girls) produced total scores of 22.5, 18, 18, 17 out of 32 all of which would have placed them comfortably in C1. Even in E1 which had relatively high mean scores had two low scores of 14 and 18.

### 8.7 Gender differences

Mean scores for boys and girls for the post-test were used to compare the performances within each group, and to compare boys in the control groups with boys in the experimental groups, and girls in the control groups with girls in the experimental groups. Pre-test scores for boys and girls were used in attempt

to assess whether the experimental or control group teaching had appeared to have had different effects on the way in which boys and girls had been able to understand the concepts being developed.

All marks are converted to percentages for comparisons between the pre-test and post-test, but given out of 32 in calculations for the analysis of variance between boys and girls for the post test.

E1 and C1

E1 Pre-test: boys mean score 44      Girls mean score 33

Post-test: boys mean score 75      Girls mean score 82

C1 Pre-test: boys mean score 44      Girls mean score 34

Post-test: boys mean score 70      Girls mean score 61

Though there is no significant difference between the boys and girls scores for either group in the pre-test or the post-test, it can be seen that for the control group the difference in favour of the boys in the pre-test has been maintained in the post-test, whereas in the experimental group the girls made a relatively greater improvement from pre-test to post-test. Moreover, of the seven pupils who showed the greatest mastery of the topic, scoring 31 or more out of 32, only one was a boy. When comparing results for boys in E1 against boys in C1 it can be seen that the boys in E1 scored more marks but this difference of 5 marks in the mean is not significant. However, the null hypothesis, that there will be no difference in the



performance of girls in the experimental group and girls in the control group leads to the following analysis:

	d.f	Sum of Squares	V.E.
Between	1	382.7	382.7
Within	31	656.6	21.2
Total	32	1039.3	

$F=18.07$  which is significant ( $p < 0.05$ ) and the null hypothesis is rejected.

E2 and C2

E2 Pre-test:	boys mean score 18	Girls mean score 18
Post-test:	boys mean score 43	Girls mean score 57

C2 Pre-test:	boys mean score 22	Girls mean score 17
Post-test:	boys mean score 50	Girls mean score 31

It can be seen that the superiority of the boys in C2 in the pre-test was actually increased in post-test 1, whereas in E2 the girls improved more than the boys. When a two tailed analysis of variance is applied to the two post-test results, with the null hypothesis that there will be no difference in scores between the boys and the girls in each group the following results are obtained:

E2	d.f	Sum of squares	V.E.
between	1	125.705	125.705
within	24	625.42	26.059
total	25	751.125	

F.ratio 4.824. The null hypothesis is rejected ( $p < 0.05$ )

C2	d.f	sum of squares	V.E.
between	1	250.353	250.353
within	27	1145.181	42.414
total	28	1395.534	

F ratio 5.903. The null hypothesis is rejected ( $p < 0.05$ )

The girls in E2 performed significantly better than the boys, whereas the boys in C2 performed significantly better than the girls

The boys in C2 scored slightly higher than the boys in E2 but the remarkable and highly significant difference occurs in the scores between the girls in E2 and the girls in C2, with the former having a mean score which is 26.1% above the latter.

The second major hypothesis was with respect to a comparison of performance between the girls in the control group and the girls in the experimental group, and between the boys in the control group and boys in the experimental group.

The null hypothesis that there will be no difference in performance between boys in the control group and boys in the experimental group is accepted (mean for experimental group 13.9, for control group 15.9).

The null hypothesis that there will no difference in performance between girls in the control group and girls in the experimental group leads to the following analysis:

	d.f	Sum of Squares	V.E.
between	1	452.572	452.572
within	24	833.274	34.72
total	25	1285.846	

F ratio 13.035, significant ( $p < 0.05$ ). The null hypothesis is rejected.

As with E1, C1, there is a significant difference between the performances of the girls in favour of the experimental group, but no significant difference between the performances of the boys.

E3 and C3

E3 Pre- test: boys mean score 22      Girls mean score 20

Post-test: boys mean score 56      Girls mean score 46

C3 Pre -test: boys mean score 24.      Girls mean score 18

Post-test: boys mean score 24      Girls mean score 36

At this level, there is no significant difference between the performances of the boys and girls within each group, for either group. It can be seen that the post-test mean for the girls in C3 is now higher than the post-test mean for the boys in C3 whereas in the pre-test the reverse was true. However, it is the very poor performance of the boys in this group which is perhaps the salient point, for the girls in C3 still scored fewer marks than the girls in E3 in both sections of the test. In E3 the boys maintained their better performance in the pre-test into the post-test.



A comparison of the scores of girls in the control group and girls in the experimental group, leads to acceptance of the null hypothesis, that is no difference in performance between the two samples, whereas an analysis of the boys' results shows:

	d.f.	Sum of Squares	V.E.
between	1	482.299	482.299
within	17	317.727	18.69
total	18	800.026	

F ratio of 25.805, which is significant ( $p < 0.05$ )

The null hypothesis, that there is no difference in performance between the boys in the control group and the boys in the experimental group is rejected. For these groups the girls in the experimental group performed better than the girls in the control group, but not significantly so, whereas the boys in the experimental group performed significantly better than the boys in the control group.

E4 and C4

No pre-test was given to this group

E4 Post-test: boys mean score 37%      Girls mean score 34%

C4 Post-test: boys mean score 15%      Girls mean score 8.%

Though no significant difference occurs between boys and girls within either of the groups the girls in E4 can be seen to have a much closer score to the boys in the group than the girls in C4 have to the boys in that group.

A comparison of scores between the groups for boys gives:

	d.f	Sum of Squares	V.E.
between	1	135.552	135.552
within	9	207.175	23.019
total	10	342.727	

F ratio 5.889. This is significant ( $p < 0.05$ ) and the null hypothesis that there will be no difference in performance between the boys in the control group and the boys in the experimental group is rejected.

A comparison of the girls scores gives:

	d.f	Sum of squares	V.E.
between	1	272.064	272.064
within	15	360.436	24.028
total	16	632.500	

F ratio 11.32. This is significant ( $p < 0.05$ ) and the null hypothesis, that there will be no difference between the performance of the girls in the control group and the girls in the experimental group is rejected.

These results provide evidence that, at least at the higher ability ranges, when exposed to a teaching approach relying on the interrelation of visual imagery to numerically coded information and algebraic routines, the girls appear to have gained more than boys the same group. In particular, the comparisons made between girls in E1 and E2 with girls in the equivalent control groups indicates that girls did not respond well to teaching based around applying trigonometrical procedures to problems, then attempting to reflect on these in order to understand geometrical properties.

### 8.8 The matched pairs

The matched pairs provide an opportunity for comparing experimental and control group scores, and to compare girls and boys scores across the ability range using a non-parametric test. The results are as follows.

All pairs

Pair Code	Score (E)	Score (C)	Difference	Rank
A1B	23	17	+ 6	+5.5
B1B	20	21	- 1	-1
C1B	27	12	+ 15	+15
D1G	32	20	+ 12	+13.5
E1G	31	20	+ 11	+10.5
F1G	26	15	+ 11	+10.5
G2B	20	8	+ 12	+13.5
H2B	18	15	+ 3	+3.5
I2B	10	18	- 8	-7
J2G	22	1	+ 21	+16
K2G	21	19	+ 2	+2
L2G	24	14	+ 10	+8
M3B	15	4	+ 11	+10.5
N3B	18	15	+ 3	+3.5
O3G	8	14	- 6	-5.5
P3G	22	11	+ 11	+10.5

Sum of positive ranks 122.5      Sum of negative ranks 13.5

The null hypothesis, that there is no difference between the students who experienced the experimental treatment and those who received the control treatment is rejected ( $p \leq 0.05$ ).



In fact this result is significant for a two tailed test at  $p < 0.01$

When the girls and boys are considered separately, the difference in ranks are as follows:

Girls		Boys	
Pair	Rank	Pair	Rank
D1G	+7	A1B	+4
E1G	+5	B1B	-1
F1G	+5	C1B	+8
J2G	+8	G2B	+7
K2G	+1	H2B	+2.5
L2G	+3	I2B	-5
O3G	-2	M3B	+6
P3G	+5	N3B	+2.5

For the pairs of girls: Sum of positive ranks 34. Sum of negative ranks 2.

The null hypothesis, that there is no difference in performance between girls receiving the experimental treatment and those receiving the control treatment is rejected ( $p < 0.05$ )

For the pairs of boys: Sum of positive ranks 30. Sum of Negative ranks 6. This is not significant ( $p < 0.05$ ) for a two tailed test, and the null hypothesis, that there is no difference in performance between the boys receiving the experimental treatment and the boys receiving the control treatment is accepted. The poor performance of one boy in E2 (pair I2B) is enough to make the sum of the negative ranks too high by 2, for the significance level to be reached.

The pairs chosen for this test do not include members of E4 and C4, where a considerable difference in performance was evident.

### 8.9. Summary of results and conclusions

The results of post-test 2 show that for all pairs of groups the experimental groups performed significantly better in section A of the test than did the control groups.

For the second section of the test, representing questions favoured by the teachers of the control groups as being the strength of their teaching, employing well practised procedures, the groups E1 and E2 performed in a very similar way to the equivalent control groups. No disadvantage in answering these questions had accrued to the experimental groups by spending much of their teaching time developing an understanding based on the interrelation of the computer based imagery and the algebraic or numerical representation.

For pupils in the lower ability groups, the experimental approach provided a significantly better performance in both sections of the test. In particular the results for E4 and C4 show that whereas no member of the control group was able to score highly in either section of the test the experimental group performed surprisingly well on the first section and four pupils were able to score highly enough on both sections of the test to place them on similar scores to members of C1.

This facility for some pupils who have been considered unable to comprehend mathematics of this nature to respond

surprisingly well to a computer graphics approach, has been recorded in the earlier part of this thesis as a phenomenon observed in earlier research. (Blackett N, 1987). It has some implications, perhaps, for any developments being made to encourage children considered to be 'non-academic' to carry on studying mathematics.

The matched pairs test supported the results of the analysis of variance with a significant difference in favour of the experimental subjects being evident for total scores over both sections.

In terms of their success in relating visual imagery to numerically encoded information as a basis for developing a conceptual understanding of trigonometry, the girls in E1 and E2 performed much better than boys in the same groups, in the case of C2 significantly so, whereas the girls in C1 and E1 maintained the inferior performance suggested by the pre-test. When girls and boys are considered separately, the following pattern emerges.

E1/C1 Girls sig. difference in favour of E1

Boys no sig. difference.

E2/C2 Girls sig. difference in favour of E2

Boys no sig. difference.

E3/C3 Girls no sig. difference

Boys sig. difference in favour of E3

E4/C4 Boys sig. difference in favour of E4

Girls sig. difference in favour of E4.



The matched pairs test supported these findings, with the girls in the experimental sample performing significantly better than the girls in the control sample, but the boys in the experimental sample showing a difference in performance to the boys in the control groups which was not significant.

The results of the pre-test, and post-test together are summarised in the table below. 'Sig' is used to denote a significant difference in the results.

Group	Pre-Test	Post-Test 1		
		A	B	Total
E1	37	78	81	79
C1	38	47	80	64
		sig	----	sig
E2	18	56	45	51
C2	20	38	46	42
		sig	----	sig
E3	20	52	47	50
C3	21	28	31	30
		sig	sig	sig
E4	not given	48	22	35
C4	not given	17	5	11
		sig	sig	sig

## 9. Post-Test 2 Results

### 9.1 Preamble

When the first post-test had been marked they were returned to the teachers of the control groups who discussed the results. No comparisons had been made with other groups at this time. The students checked the marks and the teachers were asked for their perceptions of the marks obtained by their classes by the researcher.

Mr.A and Mr.B. were not dismayed by the results feeling that the relatively poor performances on section A of the paper was owing to a lack of rehearsal and reflection on some of the results they had established in the course of their lessons. They both thought that weaknesses could be addressed before the second test was given, and it was agreed that the classical research pattern of:

pre-test → treatment → post-test → delayed post-test

was unsuitable in this case, and wished to spend a short time emphasising some of the points they thought had been poorly understood, between the two post-tests. In this way the control groups had an advantage over the experimental groups, for the researcher did not teach any more lessons on trigonometry but gave the papers back to the groups in the last ten minutes of lessons which were being taken by their normal class teacher.

Mr.C. the teacher of C3 was more concerned. He felt that there was a distinct difference between what was being asked on section A of the test and what the students had gained from his teaching. He expressed his feelings in these terms,

"I can see what you are getting at in that first section, you are trying to see how much they understand about trigonometry, but the pupils in this group don't want to be taught these kind of things, they want to get on and find the answers. Even if I tried to take some time getting them to think about that kind of thing they wouldn't like it, they wouldn't take much notice."

Mr.C was expressing in his own terms the problems associated with children's expectation of mathematics being easily performed procedures, which could make unnecessary the efforts associated with conceptual understanding. The researcher had experienced the same phenomenon with some of the boys in E2 and was aware of the effect it could have on the confidence of the teacher.

Mr.D. the class teacher of C4 was dismayed at the performance of his group and now believed that trigonometry was too difficult a topic to be taught to these pupils. though he was interested in the results of E4 and believed that he would have been more successful with the same approach. He did not wish the group to be embarrassed by another test in which they would inevitably be unsuccessful and, with the agreement of the head of department it was



accepted that though E4 would attempt the second post-test, C4 would take no further part in the research.

## 9.2 The test

The test itself was designed in the same way as post-test 1, with section A consisting of short questions aiming to test the understanding of underlying concepts, and section B aiming to test the application of procedures to more typical routine questions and problems associated with trigonometry.

It had been assumed, falsely as it turned out, that the Q1, on post-test 1, relating to the very basic concept that similar triangles had the same angles, would be too trivial to include in the second test. Consequently, this was not tested again. Upon reflection this may have been an omission, for there were many pupils, particularly in C3 and C4, who were unsure about this, and it may have been useful to gauge how successful Mr.C. had been in drawing their attention to this.

The questions in section A were intended to test:

1. Whether the students had arrived at an understanding of the meaning of the sine of an angle, in particular whether there would be any cognitive conflict if an incorrect attempt to apply a previously learnt definition contradicted the visualisation of the geometry associated with the diagram.

2. Whether the students could use the understanding of sine of an angle to deduce true statements about a pair of right angled triangles.
3. Whether the students understood that the cosine of an angle could not be greater than 1 but that there is a value for which  $\sin x = \cos x$ .
4. Whether the students are confident in their understanding of sine and cosine to be flexible in their thinking. In this case to realise that the sine of  $21^\circ$  can be found by using the cosine of  $69^\circ$
5. Whether the students could identify true statements about a right angled triangle drawn in a different orientation from typical diagrams.
6. Whether the students could use the given information to draw right angled triangle, recognising that if  $\sin A = \cos A$  then  $A$  must be  $45^\circ$ .

Section B consists of three questions where the student was expected to show how (s)he could calculate missing sides or angles in right angled triangles, and two questions given in the form of word problems. The situations described by these problems should be familiar to the students.

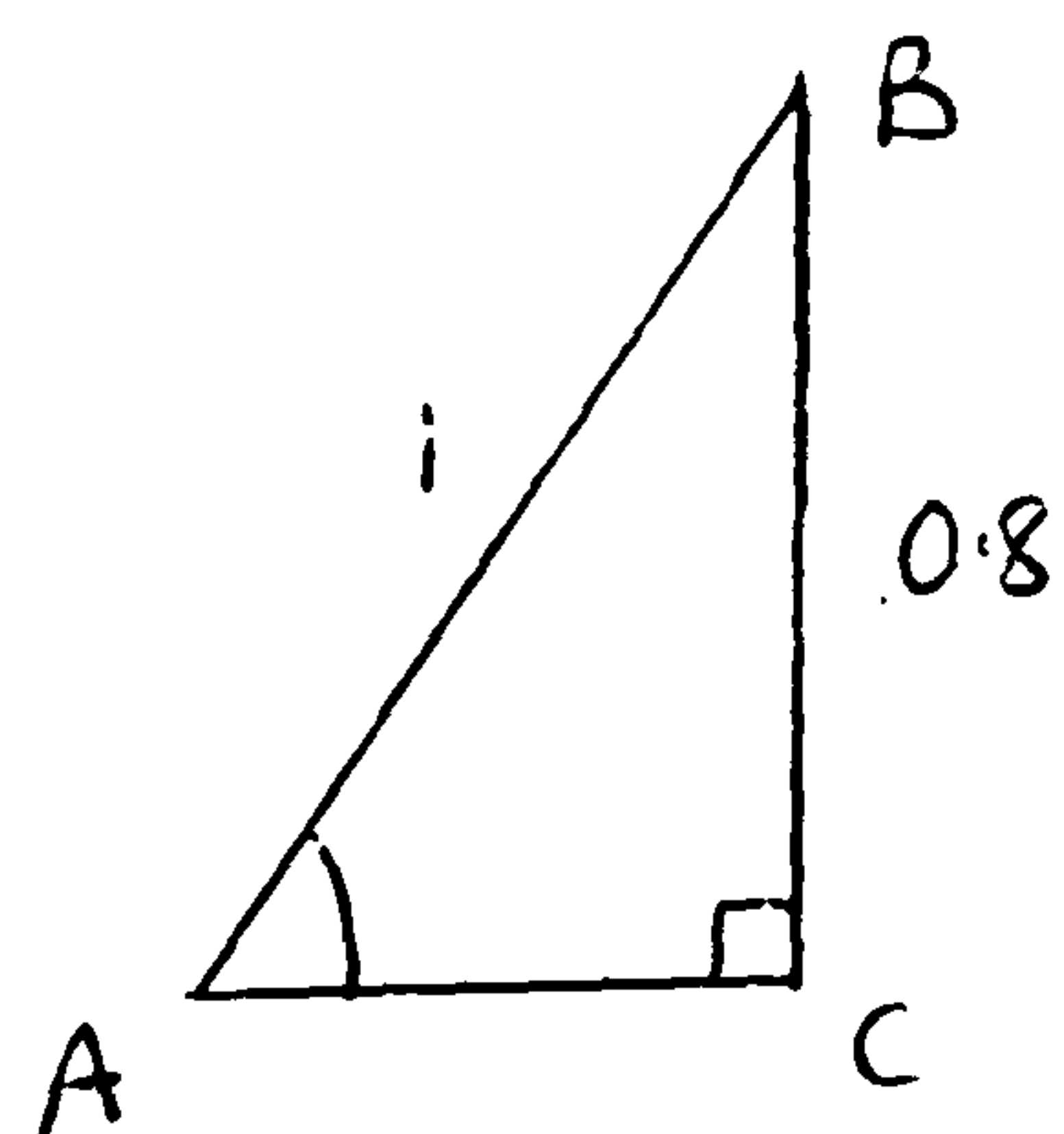
As with post-test 1, teachers were allowed to help with difficulties occurring when reading the question or interpreting the notation.

The question paper in full is as follows.

## Post-Test 2

Part A. You will not require calculators for this section.

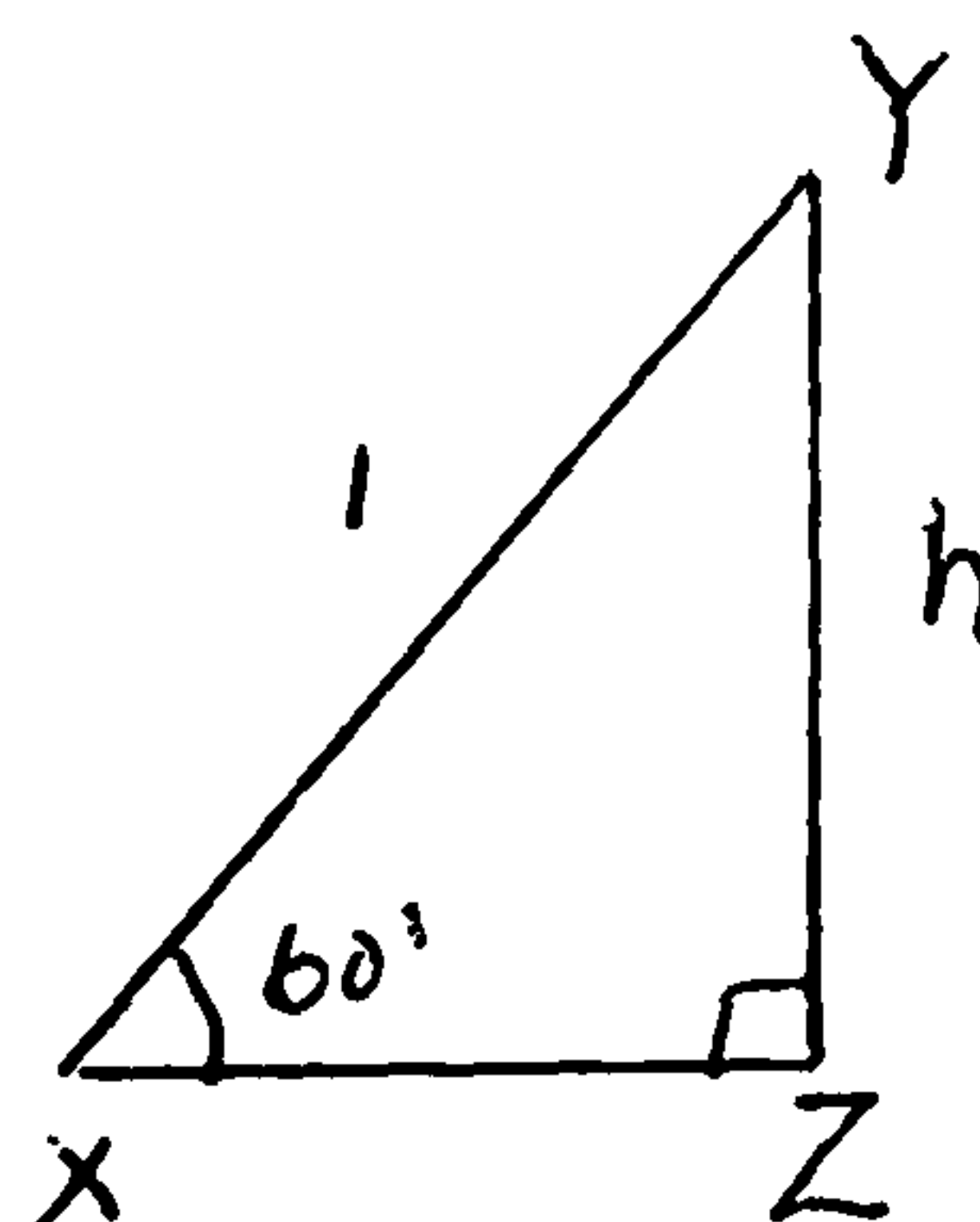
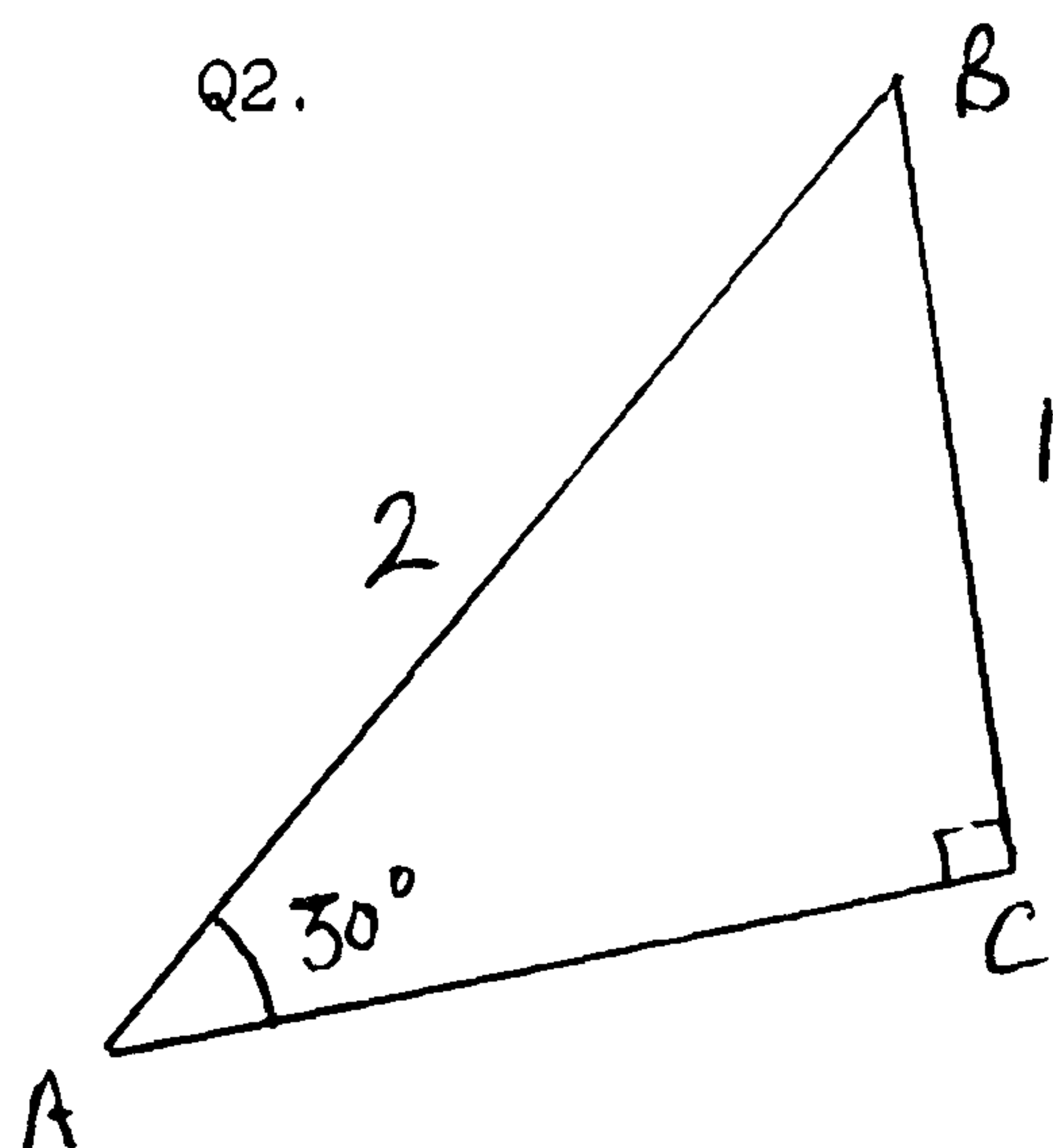
Q1.



The sine of A is

- a) 90
- b) 1.6
- c)  $1/0.8$
- d) 1
- e) 0.8
- f) don't know

Q2.



Which two of these statements are true for these two triangles?

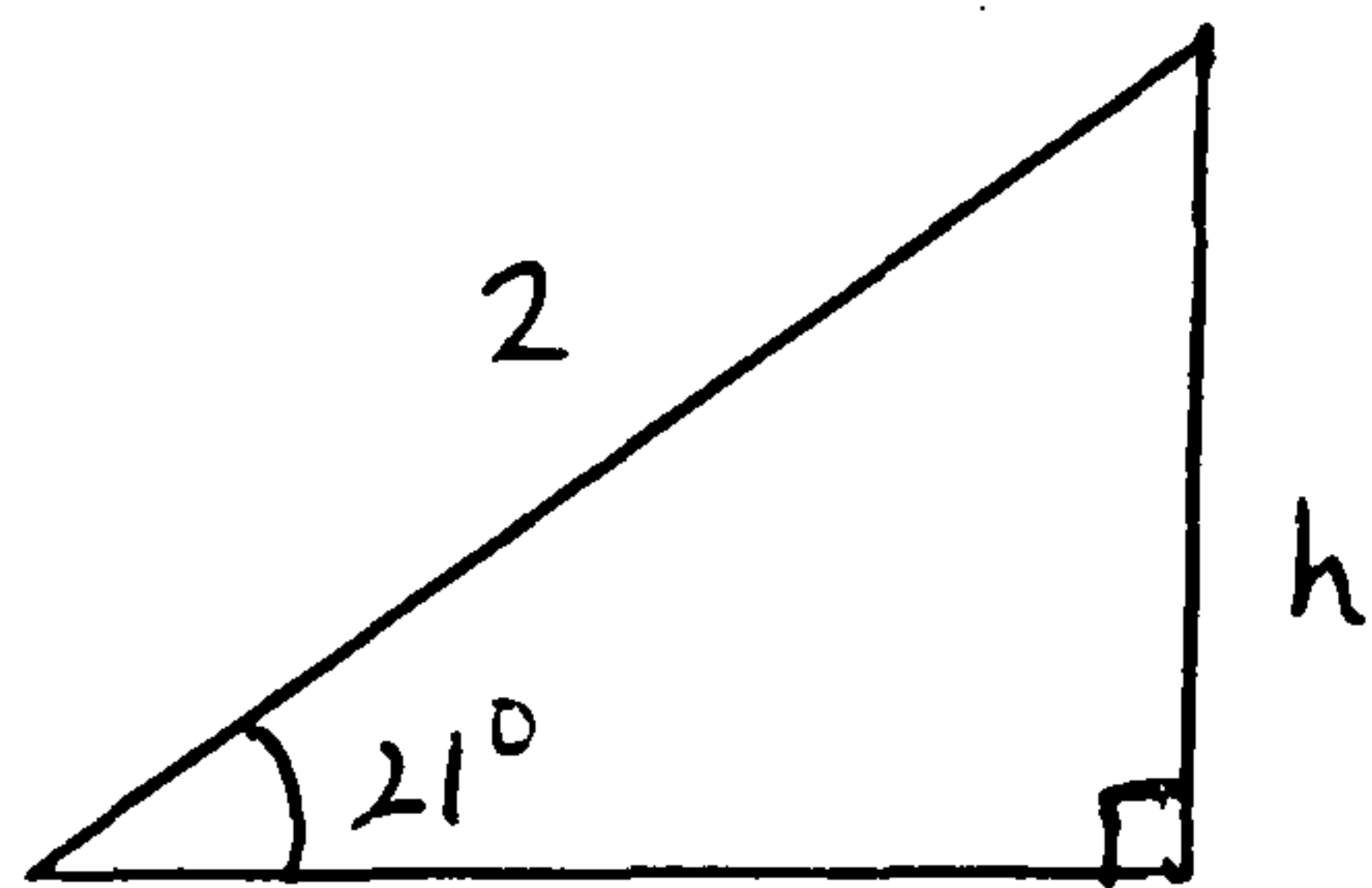
- a)  $h = \sin 90$
- b)  $AC = 2 \cos 30$
- c)  $AC = 4XZ$
- d)  $h = 1$
- e)  $h = 2 \sin 30$
- f)  $h = \sin 60$
- g)  $BC = AC$



Q3. Which one of these statements can not be true for any right angled triangle with an angle of  $x$

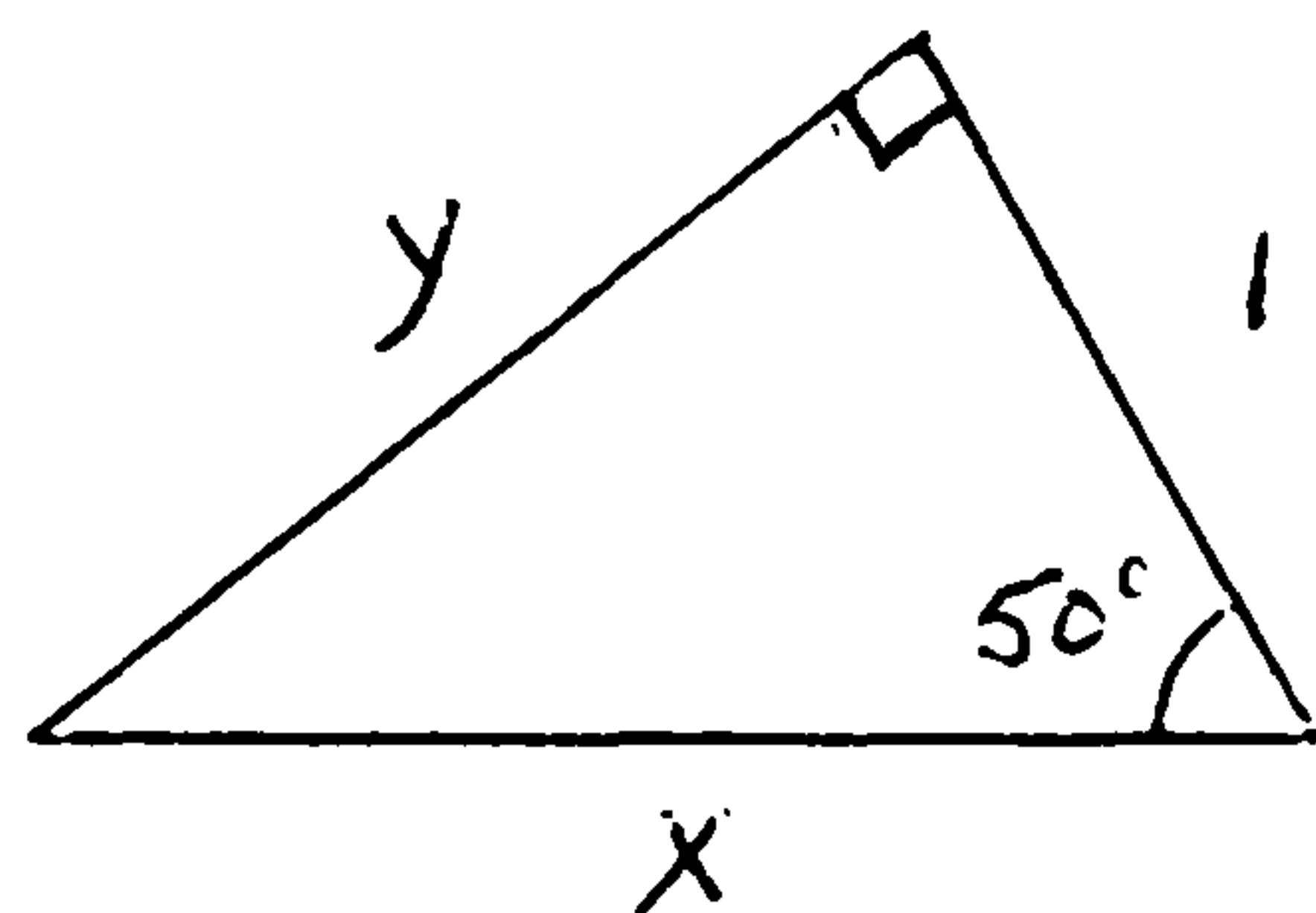
- a)  $\cos x = 0.1$       b)  $\cos x = 2.1$       c)  $\sin x = 0.001$   
 d)  $\sin x = \cos x$       e)  $\tan x = 24.2$

Q4.



Explain how you could calculate  $h$  if you were unable to calculate, or find  $\sin 21$ . (For instance if the SIN key on your calculator was broken but everything else was working correctly)

Q5



Which two of these statements are true?

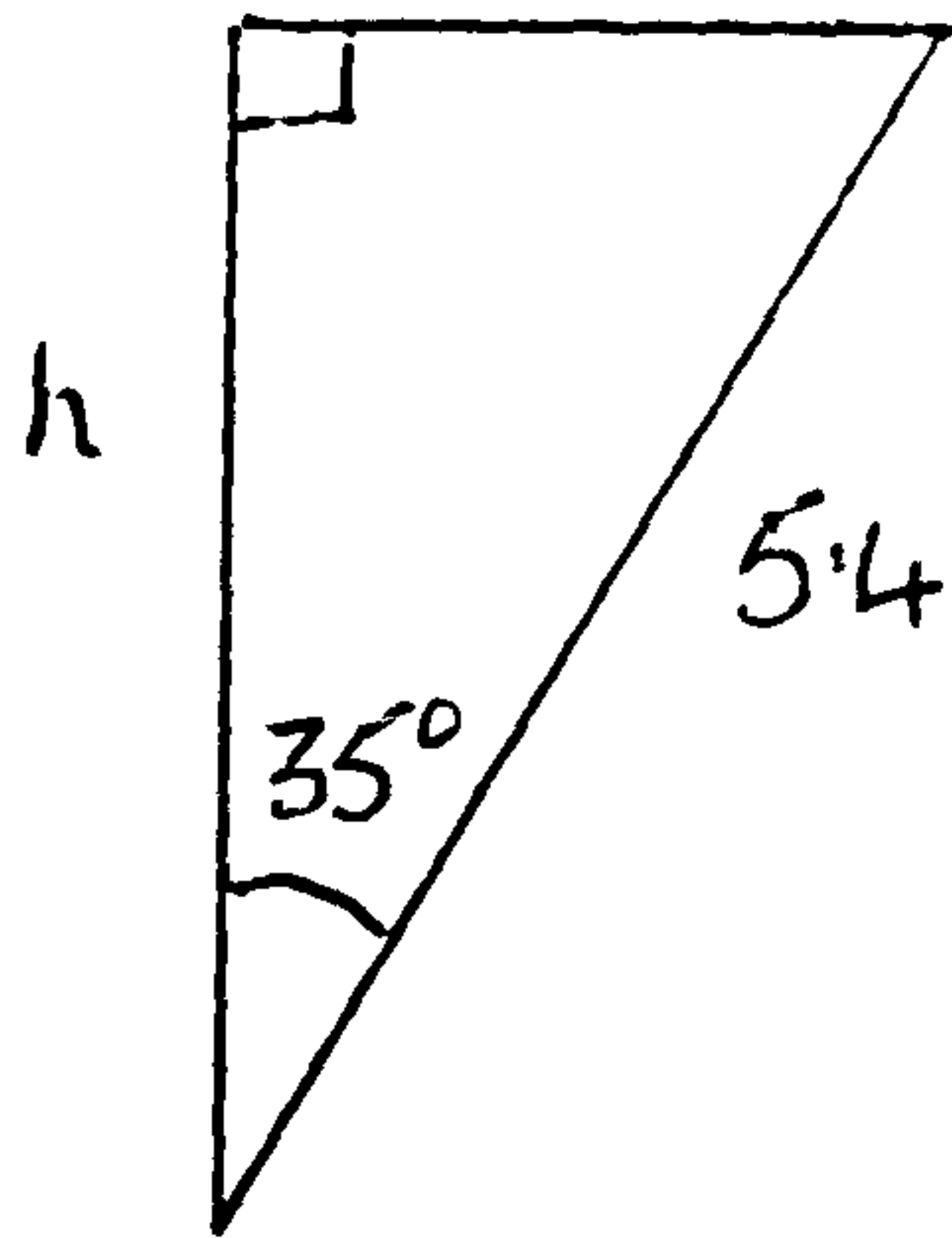
- a)  $y = \cos 50$   
 b)  $y = \sin 50$   
 c)  $y = x \sin 50$   
 d)  $y = 1$   
 e)  $y = x$   
 f)  $y = \tan 50$   
 g)  $x = \sin 90$

Q6. Draw a sketch of a triangle ABC, where angle  $C = 90$ ,  $AB = 4$  and  $\sin A = \cos A$ . Use a ruler and mark the values of angle A and angle B

Note: This means the sine of angle A is the same as the cosine of angle A.

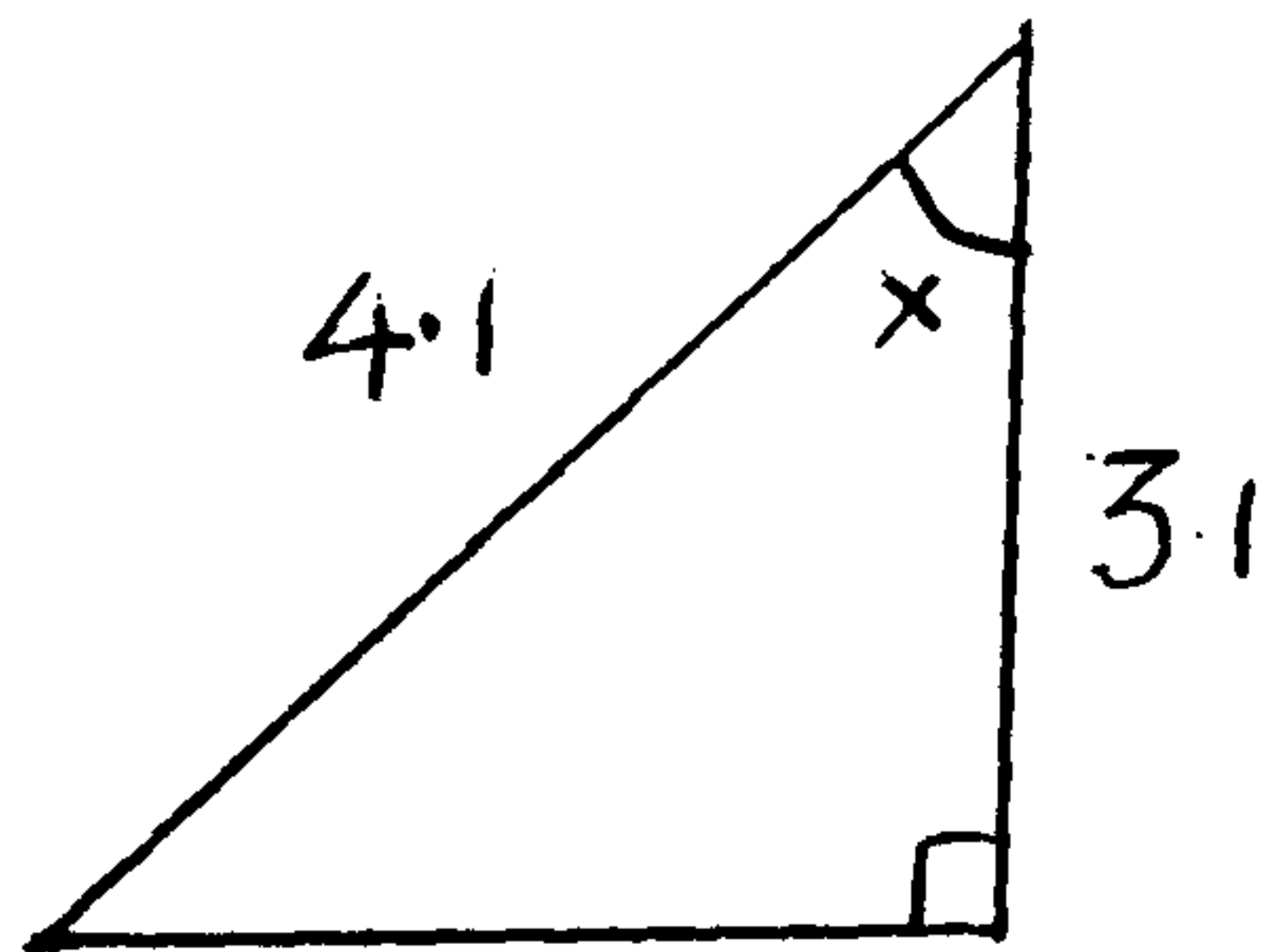
Part B.

Q7.



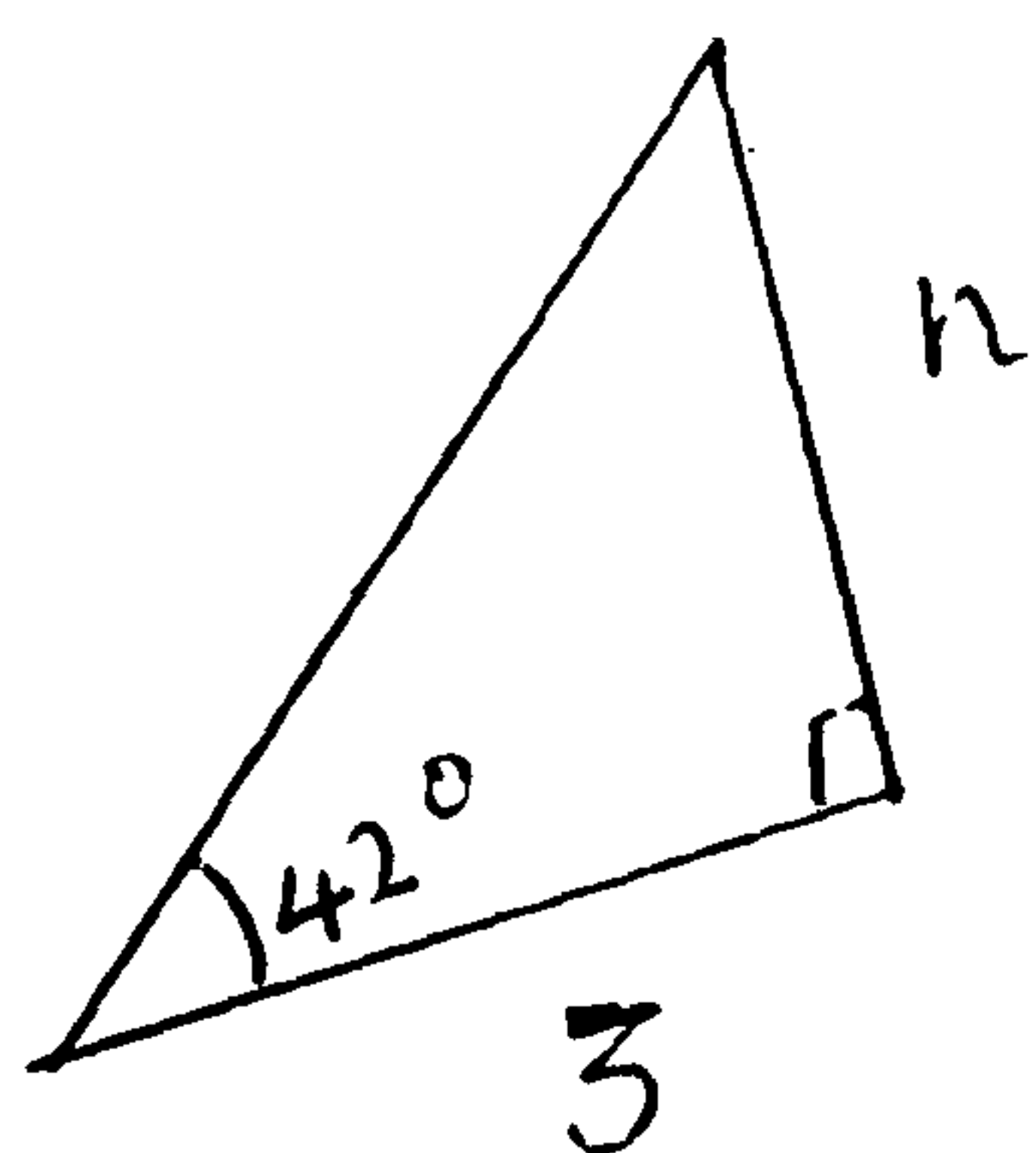
Find the length  $h$ . (Write an expression to show how you are calculating  $h$ )

Q8.



Calculate the angle  $x$ . (Write an expression first)

Q9.



Calculate  $h$ . (Write an expression first).

Q10. A train moves along a railway line which is inclined at  $10^\circ$  to the horizontal. (The train is climbing a hill where the railway line makes an angle of 10 degrees to the horizontal). The train travels for 520m along the line. How much higher is it than when it started?

Q11. A length of wood 3.2m long leans against a vertical wall. It makes an angle of 24 with the wall.

How far away from the wall is the bottom of the wood?

The wood slips down the wall, so that it makes an angle of 34 with the wall.

How much **further** is the bottom of the wood from the wall?



### 9.3 The results for E1 and C1

In a similar way to post-test 1, each individual question was marked and recorded on a grid to enable the class scores for each question to be recorded. These grids were given to the class teachers who checked them by returning the papers to the students and examining the marks awarded for each question.

9.3.1 Comparison of responses to individual questions for E1 (n=25, 16 girls, 9 boys) and C1 (n=29, 16 girls, 13 boys, 1 absent)

Q1. A single mark given for the identification of the correct option.

E1 84% correct

C1 72% correct

The most common error, was to select 1/0.8

Q2. One mark for each of the two correct options.

E1 88% two marks, 12% one mark

C1 66% two marks, 24% one mark

Q3. Two marks for the correct identification of the statement which could not be true.

E1 68% correct

C1 24% correct

The most popular error was that  $\sin x = \cos x$  could not be true.

Q4. Two marks for a correct description of a method and one mark for some expression showing an understanding of the problem but not arriving at an expression for  $h$ .

E1 72% two marks ( $h=2\cos 69$ ) 8% one mark, 20% no marks

C1 20% two marks 23% one mark, 57% no marks

There was a very different set of responses to this question from the two groups, with 12 of C1 writing that they could not see how it could be done, and those who gained two marks involving themselves in lengthy calculations. The 18 students from E1 who gained two marks were able to express the answer in two or three statements.

Q5. One mark for each correct response

E1 64% two marks, 36% one mark

C1 24% two marks 52% one mark, 24% no marks

The different orientation of the triangle caused proved to be a greater problems for members of C1 than members of E1. Seven members of C1 included options which they had clearly not analysed or related to the drawing in this orientation, such as  $x=\sin 90$ . It is easy to see how this could be selected if the students try to apply procedures established with a limited prototype to this variation.

Q6. Three marks for a sketch of an isosceles right angled triangle ABC with the right angle at C and 45 degrees marked in angles A and B. Two marks or one mark given for part correct sketches

E1 84% three marks, 8% two marks, 8% no marks.

C1 55% three marks, 45% no marks.

Again there was marked difference in the way in which this question was attempted. Several members of C1 wrote lines of 'working out' before writing that they could not do it.

The first section was marked out of twelve and the results compared. The mean mark for E1 is 7.96 and for C1 the mean is 4.87. The analysis of variance shows a significant difference ( $p < 0.05$ ) in favour of the experimental group.

Mr.A. had attempted to point out to members of C1 where they had made mistakes on the first section of post-test 1, relating answers to his earlier teaching. However, there is no evidence that these students were better able to answer these questions because it. Q3 showed that few of the members of C1 appreciated that the cosine of an angle could not be greater than one unit, despite the efforts of Mr.A. to point out the fact.

The reorientation of the triangle in Q5 caused great problems to students trying to apply a procedure to a limited well learnt prototype, and the problems members of C1 had with Q4 and Q6 relate to the wish to apply one of the learnt procedures to every situation without reflection on the situation being represented. One girl in C1 attempted to apply procedures to Q2 and after several lines of working arrived at the two statements  $h = \sin 90$ , and



$h = \sin 60$  as the two correct statements without any evidence of an awareness of the contradiction implied.

In this section of the test 44% of E1 scored full marks.

The results for the second section, containing questions similar to those well practised by the control groups were:

Q7. Two marks for the correct solution, one mark for initial correct statement without correct solution, or incorrect expression correctly followed through.

E1 100% two marks

C1 93% two marks, 3% one mark, 4% no marks.

Both groups performed better on this question than on the equivalent question on post-test 1. This may be some indication of the high motivation to perform well displayed by both groups.

Q8. Marked in a similar way to Q7.

E1 72% two marks, 8% one mark, 20% no marks

C1 83% two marks, 3% one mark, 14% no marks

A similar result to post-test 1 with the control group capable of applying the procedure correctly.

Q9. Marked in a similar way to Q7,8

E1 76% two marks, 20% one mark, 4% no marks

C1 79% two marks 3% one mark 18% no marks.

Q10. Three marks for the correct solution, two marks for a correct interpretation of the question but with a mistake in the calculation and one mark for a correct statement indicating the change in height.

E1 80% three marks, 8% two marks, 12% one mark. 0% no marks.

C1 72% three marks, 0% two marks, 10% one mark, 18% no marks.

Q11. Three marks for a correct solution, one mark each for the correct distances of the bottom of the wood from the wall, or one mark if incorrect ratio carried through without further errors.

E1 84% three marks, 8% two marks, 8% one mark, 0% no marks

C1 66% three marks, 14% two marks, 7% one mark, 13% no marks.

This section was marked out of twelve and the marks compared using the analysis of variance, as with post-test 1. The delay between the two tests appears to have had little effect on performance, except in the case of three individuals in the control group who scored poorly in both sections of the test. The mean marks, although close, show a slight superiority in favour of the experimental group, with E1 having mean 10.64 and C1 having a mean of 9.65. Clearly there is no significant difference between these two means.

All students were encouraged to add their comments on the paper. The three students from C1 who had achieved marks of 0,2 and 5 (two girls and one boy)commented:

"my memory went completely"

"I thought I understood this but I don't"

"these questions aren't like we have done before, they are not what we have learned"

The lowest scores for E1 were 12,13 and 14.

At the other end of the range of marks, there was again a degree of mastery shown by the experimental group which was different from the control group. The percentage of students achieving above half marks can be compared in three mark bands:

Section A    7-8        9-10    11-12

E1    24%        12%    52%

C1    21%        31%    0%

Section B

E1    12%        8%       72%

C1    10%        21%       59%

Totals    16-18    19-21    22-24

E1    12%        28%       40% (32% scoring 24)

C1    14%        31%       10% (0% scoring 24)



### 9.3.2 Statistical analysis of results for E1 and C1.

The one way analysis of variance was applied to these results with null hypothesis that there will be no difference in performance between the two groups. The test will be a two tailed test, level of significance  $p < 0.05$ .

For this section, and throughout this chapter, d.f. refers to degrees of freedom and V.E. refers to variance estimate

For section A: E1 mean 9.984      s.d. 2.46

C1 mean 6.034      s.d. 3.05

	d.f.	Sum of Squares	V.E.
between	1	194.43	194.43
within	52	420.3	8.083
total	53	614.76	

F ratio 24.054.

The null hypothesis is rejected ( $p < 0.05$ )

For section B and for total scores the standard deviations for the two groups differ significantly. As the analysis of variance assumes homogeneity of the variances of the two samples the procedure is not valid. Although the test is robust for small samples which are approximately equal (Edwards A.L., 1968) it appears safer in this instance to remove the two lowest scores from C1, (both 0 for section B) which make a major contribution to the large standard deviation, treating them as aberrations. This has the

effect of raising the mean for C1 slightly, from 15.7 to 16.6.

For section B E1 mean 10.64 s.d. 1.874

C1 mean 10.37 s.d. 2.511

F ratio 0.183. This is not significant ( $p > 0.05$ ). The null hypothesis is accepted.

For total scores: E1 mean 20.48 s.d. 3.646

C1 mean 16.63 s.d. 4.347

	d.f.	Sum of Squares	V.E.
between	1	192.44	192.44
within	50	842.54	16.64
total	51	1034.96	

F ratio 11.421 This is significant ( $p < 0.05$ ) and the null hypothesis is rejected.

Even when using the adjusted scores the means differ significantly in favour of the experimental group.

#### 9.4 Results for E2 and C2

9.4.1 Comparison of responses to individual questions for E2 (n=25, 13 girls, 12 boys, 1 absent) and C2 (n=26, 11 girls, 15 boys, 3 absent)

Q1. A single mark for the identification of the correct option.

E2 88% correct

C2 50% correct-the major error being the selection of  $1/0.8$  as the sine of  $x$ .

Q2. One mark for each correct statement.

E2 64% two marks, 20% one mark, 16% no marks

C2 31% two marks, 58% one mark, 11% no marks

Q3. E2 56% correct

C2 23% correct

Q4. E2 20% two marks, 48% one mark, 32% no marks.

C2 12% two marks, 15% one mark, 73% no marks

The great majority of C2 were unable to see any way of approaching this question, though several did write down  $h/2 = \sin 21$  and attempted to manipulate this before giving up.

Q5. E2 20% two marks, 60% one mark, 20% no marks

C2 12% two marks, 15% one mark, 73% no marks

Q6. E2 80% three marks, 4% two marks, 12% one mark, 4% no marks

C2 31% three marks, 4% two marks, 31% one mark, 34% no marks.

There was a marked difference between the two groups in the way this question was attempted. A number of members of C2



were unable to draw any sketch at all. some writing messages to the effect that they could not see how to do it, others had written some algebraic statements which had added nothing to their understanding. The members of E2 were much more successful in their interpretation of the question and in the accuracy of their sketches.

The analysis of these results indicate that Mr.B had not been successful in trying to bring about a more conceptual understanding of these ideas by reflecting on mistakes made in post-test 1. The mean scores differ significantly in favour of the experimental group.

The results for the second section were:

Q7. E2 64% two marks, 30% one mark, 6% no marks

C2 46% two marks, 31% one mark, 25% no marks

Q8. E2 28% two marks, 36% one mark, 36% no marks

C2 19% two marks, 15% one mark, 66% no marks

Q9. E2 20% two marks, 56% one mark, 24% no marks

C2 38% two marks, 27% one mark, 35% no marks

Q10. E2 56% three marks, 8% two marks, 24% one mark, 12% no marks.

C2 27% three marks, 4% two marks, 12% one mark, 57% no marks.

Again, a clear difference between the two groups appears here, in that the number of pupils from C2 who were unable to attempt this question greatly exceed the number who were unable to attempt it from E2.

Q11. E2 24% three marks, 20% two marks, 32% one mark, 24% no marks.

C2 8% three marks, 23% two marks, 12% one mark, 57% no marks.

Comparing the results of this section with the equivalent section in post-test 1, reveals that the differences emerging in the first test have been enlarged in the second test. The experimental group are now performing better in all parts of this section, which may have been expected to be the strength of the control group, except in the question on the tangent, where some members of E2 used the incorrect ratio. In particular the ability to solve problems requiring a diagram from a word problem appears to have deteriorated in the control group

#### 9.4.2 Statistical analysis of results for E2 and C2.

The one absentee from E2 does not make it difficult to compare the results of post-test 2 to post test 1 for this group, especially as her score in post-test 1 was close to the mean. The mean mark of the three absentees from C2 was

38.4, whereas the mean mark for the control group as a whole was 41.9%. It seems reasonable to assume that if they had been able to sit the test their marks would not have raised the mean of the group to any significant effect.

For section A: E2 mean 7.96 s.d. 2.029

C2 mean 4.92 s.d. 2.479

Using one way analysis of variance for the two samples with the null hypothesis that there is no difference in performance between the two groups:

	d.f	Sum of Squares	V.E.
between	1	117.547	117.547
within	49	262.806	5.363
total	50	380.353	

F ratio 21.917. This is significant ( $p < 0.05$ ). The null hypothesis is rejected.

For section B: E2 mean 6.684 s.d. 3.03

C2 mean 4.62 s.d. 3.659

	d.f.	Sum of Squares	V.E.
between	1	63.074	63.074
within	49	577.514	11.786
total	50	640.588	

F ratio 5.352. This is significant ( $p < 0.05$ ). The null hypothesis is rejected.



For totals: E2 mean 14.84      s.d. 4.388

C2 mean 9.577      s.d. 5.115

	d.f.	Sum of Squares	V.E.
between	1	353.039	353.039
within	49	1161.706	23.708
total	50	1514.745	

F ratio 14.891. This is significant ( $p < 0.05$ ). the null hypothesis is rejected.

The significant difference recorded for section A in post-test 1 has been repeated in this test but in addition the small superiority in favour of the control group for section B in post-test 1 has changed to a statistically significant superiority in favour of the experimental group in post-test 2.

The efforts of Mr.B. to increase the understanding of trigonometry by reflecting on the procedures they had practised failed to increase the performance of the group in section A. After a time gap of six weeks there has also been a marked decrease in the ability to answer these procedures in the section B of the test.

## 9.5 The Results for E3 and C3

9.5.1 Comparison of responses to individual questions for E3 (n=21, 13 girls, 8 boys) and C3 (n=21, 10 girls, 11 boys)

Q1 E3 62% correct

C3 14% correct

This result highlights the reliance of the control group members on visually cued procedures. Most of the group assumed that  $1/0.8$  was the correct response, in line with the visual form of the definition they had met.

Q2. E3 48% two marks, 43% one mark, 9% no marks

C3 14% two marks, 52% one mark, 34%no marks

Q3. E3 24% two marks

C3 0% two marks

Q4. E3 5% two marks, 24% one mark, 61% no marks

C3 0% no marks, 5% no marks, 95% no marks

Q5. E3 10% two marks, 43% one mark, 47% no marks

C3 5% two marks, 33% one mark, 62% no marks

Q6. E3 33% three marks, 5% two marks, 24% one mark, 38% no marks

C3 24% three marks, 19% two marks, 24% one mark, 33% no marks.

The mean marks for this section show a significant difference in favour of the experimental group.

The marks for the second section were as follows:

Q7. E3 43% two marks, 43% one mark, 14% no marks

C3 33% two marks, 14% one mark, 53% no marks

Q8. E3 14% two marks, 10% one mark, 76% no marks

C3 10% two marks, 29% one mark, 61% no marks

Both groups show a decline in the ability to find a missing angle in a right angled triangle, though the practice undertaken by the control group has not proved to be an advantage.

Q9. E3 29% two marks, 33% one mark, 38% no marks

C3 14% two marks, 29% one mark, 57% no marks

The problems the experimental group had in dealing with the tangent are still evident to some extent, but the control group, at this stage were having even more difficulty with this question.

Q10. E3 24% three marks, 10% two marks, 24% one mark, 42% no marks.

C3 14% three marks, 0% two marks, 19% one mark, 67% no marks.

Q11. E3 10% three marks, 24% two marks, 5% one mark, 61% no marks.

C3 5% three marks, 10% two marks, 10% one mark, 75% no marks.



Both groups found this section of the test difficult, but the control group had a much sharper decline in their ability to answer the kind of questions they had been familiar with.

#### 9.5.2 Statistical analysis of the results for E3 and C3.

Applying one way analysis of variance with the null hypothesis that there is no difference performance between the two groups for section A, section B and total scores gives the following analysis.

For section A: E3 mean 4.762      s.d. 2.287  
                                          C3 mean 2.714      s.d. 1.75

	d.f.	Sum of Squares	V.E.
between	1	44.024	44.024
within	40	174.095	4.352
total	41	218.119	

F ratio 10.115. This significant ( $p < 0.05$ ). The null hypothesis is rejected.

For section B: E3 mean 4.857      s.d. 3.413  
                                          C3 mean 2.905      s.d. 2.18

Although the F ratio for the two variances suggest that they may differ significantly ( $p < 0.05$ ), the analysis of variance is still appropriate being quite insensitive to

heterogeneity of variance when the number in each sample is the same.

	d.f.	Sum of Squares	V.E.
between	1	40.028	40.028
within	40	344.381	8.61
total	41	384.405	

F ratio 4.65. This is significant ( $p < 0.05$ ). The null hypothesis is rejected.

For total scores: E3 mean 9.619 s.d. 4.315

C3 mean 5.619 s.d. 2.99

	d.f.	Sum of Squares	V.E.
between	1	168	168
within	40	579.905	14.498
total	41	747.905	

F ratio 11.588. This is significant ( $p < 0.05$ ). The null hypothesis is rejected.

It can be seen that for these groups the significant difference in favour of the experimental group, which was evident in both sections of the post-test and for the total score has been repeated in this delayed post-test.

#### 9.6 The results for E4 (n=12, 6 boys, 6 girls)

No comparison can be made with the control group for it was decided, after the results for post-test 1 were analysed,

that the control group should not have to experience that kind of failure. Clearly the lessons had not equipped the members of C4 to deal with questions of this type. In fact, the class teacher and the head of department had concluded that trigonometry was too difficult for the group. Members of E4 were willing to take the test, and the results were as follows.

Q1. 50% correct

Q2. 33% two marks, 33% one mark, 34% no marks

Q3. 42% two marks

Q4. No marks scored. The group clearly had problems in interpreting this question.

Q5. 0% two marks, 58% one mark, 42% no marks

Q6. 25% three marks, 17% two marks, 42% one mark, 16% no marks.

The mean mark for this section was 4.42 (36.8%). This represents a result which is higher than that achieved by C3 and close to the mean mark for E3.

The marks for the second section were:

Q7. 8% two marks, 42% one mark, 50% no marks

Q8. 0% two marks, 33% one mark

Q9. 0% two marks, 42% one mark

These three questions illustrate the difficulty the group had in remembering which ratio to use in each of these cases. Those who received one mark were those who attempted to use an incorrect ratio but completed the working



correctly. They would have benefited from access to information which told them which ratio to use.

Q10. 25% three marks, no other marks awarded.

Q11. 17% three marks, 0% two marks 25% one mark.

The mean for this section is 2.67 (22%)

The mean mark for the whole test is 7 (29.2%)

The mean for this group is thus higher than that of C3.

Perhaps more telling than the mean mark is the range of individual scores: 13, 13, 13, 8, 7, 7, 6, 6, 4, 4, 3, 0.

It can be seen that the three individuals who had appeared to to work successfully with trigonometry using the computer had in fact been able to perform remarkable well.

The scores they achieved are higher than the mean mark for C2 and higher than some individual scores in C1 and E1. This despite the fact that trigonometry was considered too difficult for them.

These results support earlier findings (Blackett N., 1987) referred to in chapter 1, where pupils considered to be of poor mathematical ability have responded well to visual approach to presenting a global view of concepts rather than a serialistic, procedural approach.

## 9.7 Gender Differences

As with post-test 1, the mean scores for boys and girls in both experimental and control groups are used to compare the performances of boys and girls. For the purpose of comparing marks with the pre-test and post-test 1, the marks are given as percentages, though for the analysis of variance they are marks out of 24

### E1 and C1

E1 Pre-test	boys' mean 44	girls' mean 33
Post test 1	boys' mean 75	girls' mean 82
Post-test 2	boys' mean 72	girls' mean 92
C1 Pre-test	boys' mean 44	girls' mean 34
Post-test 1	boys' mean 70	girls' mean 61
Post test 2	boys' mean 70	girls' mean 62

The girls in E1 improved their performance in post-test 2 in comparison with the boys in the same group. The standard deviation for the girls in this group is 2.029 and that of the boys 4.178 and these vary significantly ( $p < 0.05$ ) which invalidates the analysis of variance for these unequal sample sizes. However, using the modified Student's t test suggested by Edwards (Edwards A.L., 1968, p102), the value of t (3.67) indicates a significant difference in favour of

the girls ( $p < 0.05$ ). Student's  $t$  test is equivalent to the two sample analysis of variance with  $t^2 = F$ , and this modified value of  $t$  is compared with newly established critical values which take into account the unequal sample sizes. The null hypothesis that there is no difference between the performance of the boys and the girls in this group is rejected. The experimental treatment has resulted in a significant difference in the variances for the boys and girl and significant differences in the means. The small standard deviation and high mean score for the girls is an indication of their degree of mastery, with seven of the sixteen girls achieving full marks.

There is no significant difference between the mean scores of the boys and girls in C1.

In comparing the boys in the experimental group with the boys in the control group, there is no significant difference between the mean scores, and the null hypothesis that there is no difference in performance between these two samples is accepted.

When comparing the scores of the girls in the experimental group with the scores of the girls in the control group, there is again the problem of two very low scores in the control group leading to a large standard deviation. Treating these two scores of 0 and 2 as aberrations and omitting them from the analysis raises the mean of the



control group from 62% to 70%. and leads to the following analysis of variance.

E1 mean 22            s.d. 2.029

C1 mean 16.79        s.d. 3.075

The s.d. do not differ significantly

	d.f.	Sum of Squares	V.E.
between	1	208.74	208.74
within	28	202.36	6.978
total	29	411.097	

F ratio 29.91. This significant ( $p < 0.05$ ) and the null hypothesis: that there is no difference in performance between the girls in the control group and the girls in the experimental group is rejected.

As with post-test 1, the girls in the experimental group performed significantly better than the girls in the control group, however what is different from post-test 1 is that the girls in the experimental group also performed significantly better than the boys in the same group.

**E2 and C2.**

E2 Pre-test	boys' mean 18	girls' mean 18
Post-test 1	boys' mean 43	girls' mean 57
Post-test 2	boys' mean 59	girls' mean 65

C2 Pre-test	boys' mean 22	girls' mean 17
Post-test 1	boys' mean 50	girls' mean 31
Post-test 2	boys' mean 49	girls' mean 28

There is no significant difference between the scores of the boys and the scores of the girls in E2, though the girls continue to score more highly than the boys. (In post-test 1 this difference was significant). Comparing the scores for the girls and boys in the control group gives the following analysis of variance.

boys' mean	11.67	s.d. 5.095
girls' mean	6.73	s.d. 3.519

	d.f.	Sum of Squares	V.E.
between	1	154.831	154.831
within	24	525.515	21.895
total	25	680.346	

F ratio 7.07. This is significant ( $p < 0.05$ ).

In the control group the boys' performed significantly better than the girls, as they did in post-test 1.

A comparison between the scores of boys in the control group with boys in the experimental group, using analysis of variance, shows; that whilst the mean of the experimental group is higher than the mean of the control group, the difference is not significant. The null

hypothesis, that there is no difference in performance between the boys in the control group and the boys in the experimental group is accepted. ( $p < 0.05$ ).

As in post-test 1, there is no significant difference between the performance of the boys in the experimental group and the boys in the control group

A comparison between the girls in the control group and the girls in the experimental group leads to the following analysis of variance.

E2 mean 15.53      s.d. 4.307

C2 mean 6.727      s.d. 3.518

	d.f.	Sum of Squares	V.E.
between	1	462.587	462.557
within	22	377.413	17.155
total	23	840.000	

$F=26.965$ . This is significant ( $p < 0.05$ ) and the null hypothesis, that there is no difference in performance between the girls in the experimental group and the girls in the control group is rejected.

As in post-test 1., the girls in the experimental group performed significantly better than the girls in the control group.



### E3 and C3

E3 Pre-test	boys' mean 22	girls' mean 20
Post-test 1	boys' mean 56	girls' mean 46
Post-test 2	boys' mean 38	girls' mean 41
C3 Pre-test	boys' mean 24	girls' mean 18
Post-test 1	boys' mean 24	girls' mean 36
Post-test 2	boys' mean 24	girls' mean 23

The second post-test for these two groups follows the pattern shown with the previous two pairs of groups, with the girls in the experimental group performing better than the boys in the same group, and the boys in the control group performing better than the girls in the same group.

In this case none of the comparisons between boys and girls in the same group are significantly different.

A comparison between boys in the experimental group and boys in the control group shows that although the experimental group scored more highly, the difference is not significant for a two tailed analysis of variance ( $p < 0.05$ ). The difference was significant in post-test 1.

A comparison between the girls in the experimental group and the girls in the control group leads to the following analysis of variance.

	d.f.	Sum of Squares	V.E.
between	1	119.6	119.6
within	21	324.4	15.448
total	22	444.00	

F ratio 7.742. This is significant ( $p < 0.05$ ) and the null hypothesis, that there is no difference in performance between the girls in the control group and the girls in the experimental group is rejected.

The pattern for this group now follows that of the other two pairs with a significant difference between the performance of the girls in the experimental group and girls in the control group, but no significant difference in performance between boys in the experimental group and boys in the control group.

#### **E4 and C4**

E4 Post test 1 boys' mean 36.7      girls' mean 34.1

Post-test 2 boys' mean 33.0      girls' mean 25

C4 Post-test 1 boys mean 14.7      girls' mean 8.3

The numbers in E4 make it difficult to compare boys with girls, but it can be seen that the boys performed better than the girls, but both performed better than the members of C3.

## 9.8 The matched pairs

Using the non-parametric Wilcoxon matched pairs test on the pairs identified after the pre-test and compared after post-test 1 reveals the following result.

Pair code	Score (E)	Score (C)	Difference	Rank
A1B	19	16	+3	+5
B1B	17	16	+1	+2.5
C1B	23	8	+15	+16
D1G	23	14	+9	+9.5
E1G	24	15	+9	+9.5
F1G	22	15	+7	+8
G2B	10	6	+4	+6
H2B	16	15	+1	+2.5
I2B	12	23	-11	-11.5
J2G	17	4	+13	+14
K2G	16	3	+13	+14
L2G	19	6	+13	+14
M3B	10	4	+6	+7
N3B	10	9	+1	+2.5
O3G	8	9	-1	-2.5
P3G	15	4	+11	+11.5

Sum of positive ranks 122.0    Sum of negative ranks -14

This represents a significant difference, using a two tailed Wilcoxon Matched pairs test ( $p < 0.05$ )



This test supports the findings of the analysis of variance, showing that for the whole sample, using total scores, the experimental groups performed significantly better than the control groups.

Considering the pairs of boys and girls as separate samples gives:

Girls		Boys	
Pair	Rank	Pair	Rank
D1G	+3.5	A1B	+4
E1G	+3.5	B1B	+2
F1G	+2	C1B	+8
J2G	+7	G2B	+5
K2G	+7	H2B	+2
L2G	+7	I2B	-7
O3G	-1	M3B	+6
P3G	+5	N3B	+2

Girls: Sum of positive ranks +35. Sum of negative ranks -1  
This represents a significant difference, using a two tailed test. ( $p < 0.05$ )

Boys: Sum of positive ranks +29. Sum of negative ranks -7  
This does not represent a significant difference. ( $p < 0.05$ )

The non-parametric test supports the analysis of variance, that the girls in the experimental groups performed significantly better than the girls in the control groups.

whereas, though the boys in the experimental group performed better than the boys in the control groups, the difference was not statistically significant

### 9.9 Summary

The second post-test revealed that the patterns emerging in post-test 1 were reinforced and enlarged by the 6-7 week time interval.

For E2/C2 and E3/C3, the experimental groups achieved significantly higher mean scores in part A, part B and for the total scores.

For E1/C1 the experimental group achieved a significantly higher mean score for section A and for the total score, but though the mean score for the experimental group was higher than the mean score for the control group in section B, the difference was not significant.

E4 was able to achieve higher mean scores in both sections of the test than C3, but C4 did not sit post-test 2, owing to their poor performance in post-test 1.

The non-parametric Wilcoxon matched pairs test confirmed the general superiority of the experimental groups by showing a significant difference in their favour. ( $p < 0.05$ )

The results of the two post tests can be summarised as follows. The table shows the mean scores for part A and part B of each post-test, together with the pre-test mean.

'Sig' refers to a significant difference between the mean scores.

	Pre-test	Post-test 1			Post test 2		
		A	B	tot	A	B	tot.
E1	37	78	80	79	83	89	86
C1	39	47	80	64	50	86	68
		sig	--	sig	sig	--	sig
E2	18	56	45	51	66	57	62
C2	20	38	46	42	41	39	40
		sig	--	sig	sig	sig	sig
E3	20	52	47	50	40	40	40
C3	21	28	31	30	23	24	23
		sig	sig	sig	sig	sig	sig
E4	not given	48	22	35	37	22	29
C4	not given	17	5	11	not given		
		sig	sig	sig			

It can be seen that the members of C1 have had some success in learning and applying the procedures they learnt in section B of the tests, though they were unable to reflect on these to gain an understanding of the concepts underlying trigonometry, even when their teacher pointed out the mistakes the made and attempted to focus their thoughts on these concepts.



The members of C2 had some success in applying procedures to part B of post-test 1, but were unable to reproduce them after a delay. The experimental group E2, scored more highly than the control group on both sections of post-test 2.

The members of C3 and C4 were significantly poorer than the experimental subjects in both parts of the tests.

It was argued in chapter 5 that girls may have an advantage in being taught in the experimental manner, and this was tested by comparing boys and girls in the experimental groups separately with boys and girls in the control groups, and by comparing boys and girls in the same group with each other.

In E1, E2, and E3, the girls in the experimental groups achieved higher scores than the boys in the same group, as they did in post test 1. In the control groups, C1, C2 and C3, the boys achieved higher mean scores than the girls. The largest differences occurred in E1, where the girls' achieved a significantly higher score than the boys, and in C2, where the boys achieved a significantly higher score than the girls.

The comparison between girls in the experimental groups and girls in the control groups, and between boys in the experimental groups and boys in the control groups can be

summarised in the following table. 'Sig' refers to significant difference. ( $p < 0.05$ )

	Post- test 1		Post-test 2	
	Boys	Girls	Boys	Girls
E1/C1	not sig	sig	not sig	sig
E2/C2	not sig	sig	not sig	sig
E3/C3	sig	not sig	not sig	sig
E4/C4	sig	sig	-	-

The Wilcoxon matched pairs test confirmed the superior performance of the girls in the experimental groups, showing a significant difference in their favour, whilst the difference between the boys in the experimental groups and the boys in the control groups was not-significant. ( $p < 0.05$  ).

The results indicate that girls, more than boys, are advantaged by a teaching strategy incorporating computer graphics which link visual and numerical representaion.

## 10. The Interviews.

### 10.1 Pre-amble

The interviews took place some six weeks after the final stage of the second post-test, after the Christmas vacation. The researcher withdrew individuals from their lessons to interview them in a portable classroom situated between the two halls. In all, ten pupils from the experimental groups, including two from E4, and eight from the control groups C1, C2 and C3 were interviewed. The students were matched for ability from the sample of students available to be interviewed, so that four of the eight pairs from groups 1,2 and 3 were pairs from the original set of sixteen matched pairs used in the statistical analysis. The other two pairs were of similar ability from corresponding groups, and the remaining two pupils, from E4, were interviewed in an attempt to assess whether pupils considered of too low ability to be taught trigonometry had understood the concepts, and still had access to the necessary cognitive frames after a considerable delay. The most successful students from E1, many of whom appeared to have mastered this topic were not amongst the sample selected.

The interviews were an attempt to prompt the children into talking about the way they were thinking when attempting to solve a problem or reflecting on



trigonometry. In order to prompt this sort of discussion the interviewer gave different prompts to different children, rather than use a set script and help was given if the pupils found it impossible to answer a question. In addition, it will be seen that the prompts differ from control to experimental group members in an attempt to cue the ways in which each of the interviewees were expected to learn the relevant procedures or the way in which the initial presentation of the topic was made. With this technique the interviews can not be used as assessments in the way, for instance, the A.P.U. interviews (1980) were used to assess children,

The greater success of the experimental groups, was thought to be owing to the ways in which the experimental treatment allowed the pupil to develop a conceptual base for trigonometry from the visual, geometric, presentation linked to the numerical coding. It would be hoped that children who have been taught in this way would be able to perform the necessary procedures well but, with appropriate cueing, they would be able to switch to a visualisation of the triangles in question in a way which would allow them to assess the reasonableness of answers without recourse to the computer graphics. The highest level of his skill would be represented by a greater versatility of cognition, between visualisation and numerical or algebraic procedures being controlled by the 'executive control' in the mind of the student, either

consciously as a problem solving style, or spontaneously in terms of perceiving features and creating visual imagery based upon them.

The experimental groups were encouraged, by the use of accurately drawn computer graphics, to interpret diagrams as not only a symbolic method of conveying information, where the information is analysed and processed directly into a numerical or algebraic procedure, but to use the features of the diagram to visualise more accurately the situation being represented by the diagram. The more versatile students, it was hoped, would be able to develop the ability to move between the visualisation and the procedures necessary to calculate missing sides or angles in right angled triangles with a minimum of cueing.

The post-tests showed clear evidence that the experimental approach had been successful in building a conceptual understanding of trigonometry, but it is difficult to gauge how successfully the students were able to move between imagery and procedure when answering the more routine questions. The first stage of the interview was an attempt to ascertain the ways in which pupils interpreted diagrams in answering a relatively straightforward problem, as well as ascertaining how meaningful the trigonometry they had been taught was after a period in which they had not experienced any trigonometry.



## 10.2 The organisation and initial problem.

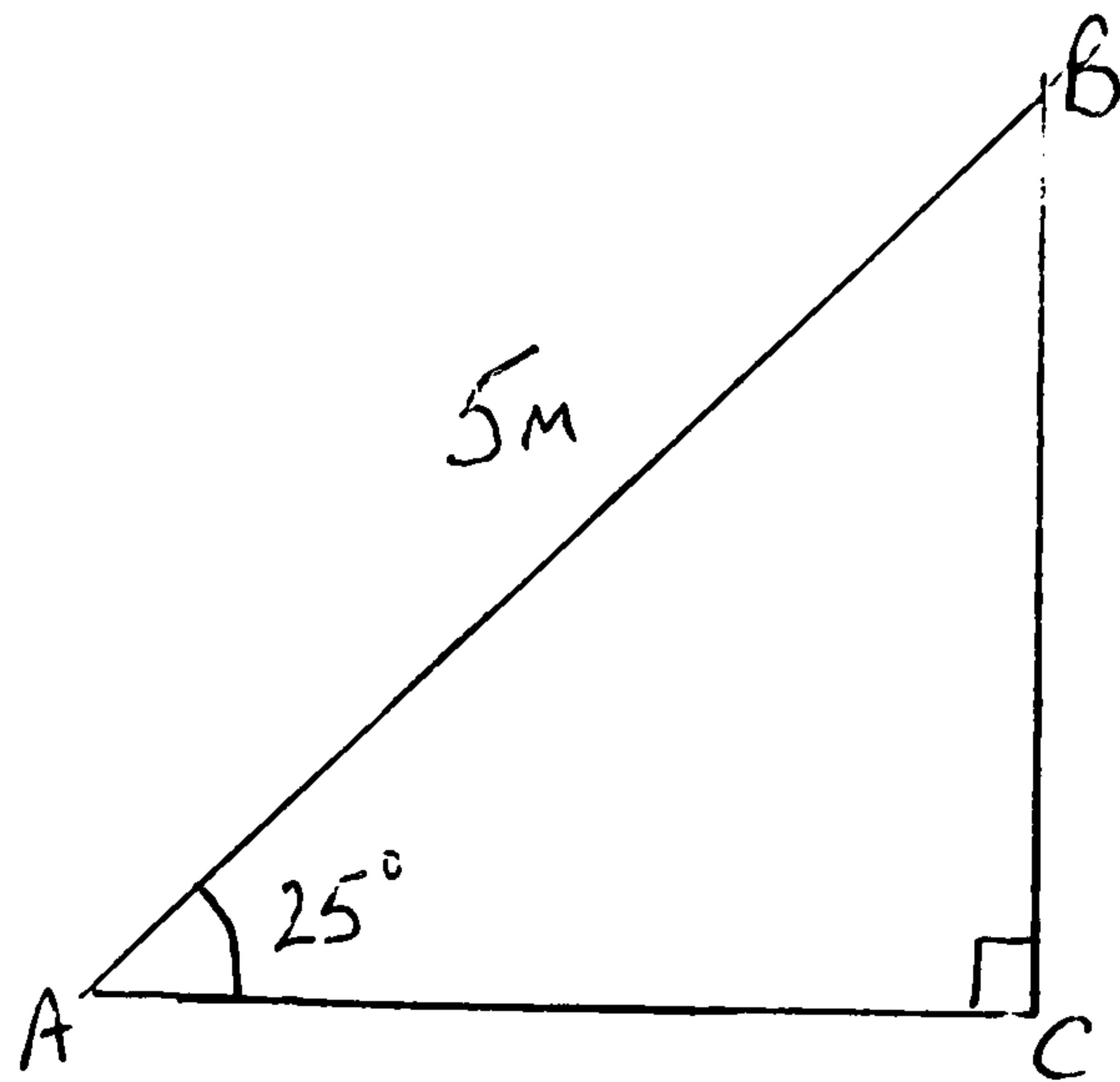
The students were withdrawn from classes such that all of the pupils in the experimental groups were interviewed on the same morning, and the control group subjects the following morning. They were interviewed one at a time, though as the members from each group were released from lessons together there was a short waiting time for some pupils in an adjacent classroom.

The interviewer and subject would be sitting on the same side of a teacher's desk, with writing material and a calculator on the desk between them. A portable cassette recorder was placed to the rear of the desk between the subject and the interviewer. On entry the pupil was informed that the researcher was still interested in how students learned trigonometry and would like the student to tell him how they attempted some questions. The tape recorder was switched on after the student was informed that it was difficult for the researcher to remember everything that was said, so a recording was necessary, but that it was only for use in the research and nothing would be used for assessment within the school.

The initial problem was designed to present a familiar situation, the ubiquitous ladder or piece of wood against a vertical wall, but the diagram was drawn in such away as to give all the necessary information to solve the



problem, yet was visually misleading. The diagram is given below.



The student would be asked to find out how far up the wall the plank of wood reached (2.11m), and assuming that this was correctly calculated, he or she would be presented with the information that before some time previously the researcher had calculated the distance of the foot of the ladder from the wall to be about 1.7m and asked to comment on the reasonableness of the answer. The value of 1.7m is consistent with the diagram if no more accurate visualisation of the situation is made, but not with the visualisation of the actual problem. Eventually the student would be asked to estimate a better answer and to go to calculate the result. The following texts were transcribed from the recordings and some attempt has been made to indicate the time taken and the tone of response. However, it is impossible to represent exactly in words the range of responses made, which included sighs, pencil tapping and laughing. Every effort was made by the researcher to make the student feel at ease, he being aware

of the nervousness felt by many of the students in this situation.

### 10.3 The interview transcriptions

Throughout these transcripts 'I' refers to interviewer, 'R' refers to the respondent. The transcripts will alternate between experimental and control subjects.

Caroline, E1:

I: Have a look at this triangle, there is our right angle, and the hypotenuse is 5m. It represents a ladder or a piece of wood leaning against this vertical wall. I want you to try and find out what this distance is here (he points do distance up the wall). How would you set about doing that?

R: I don't know.

I: have a good look at it

R: Well I would use sin 25 times 5

I And what would that tell us?

R: That would tell us BC

I: How did you know that would be BC rather than AC?

R: It was a guess, well not really, SOCHATOA (she pronounces it as a word)

I: What does that stand for?

R: Well it is meant to help you to remember  $\sin = \text{opposite over hypotenuse}$ ,  $\cos = \text{adjacent over hypotenuse}$  and  $\tan = \text{opposite over adjacent}$ .

I: Well it helped you to remember

R: she laughs

I: can you use the calculator to find BC?

R: (after a short time with the calculator). It is 2.1m

I: Did you round your answer?

R: Yes to 1 decimal place, is that all right?

I: That is fine, well done. Now suppose I told you that before you came into the room I had worked out AC to be 1.7m, what would you say to that?

R: (With no hesitation) That would be wrong because if the hypotenuse is 5m and the angle is 25 degrees then that (she points to AC) will be bigger than that (she points to BC).

I: That is true, can you write an expression for BC.?

R: Is it  $\cos 25$  times 5?

I: That is it, can you write it down?

R: (she writes  $25 \cos x 5$ , which is the order of operations using the calculator, but quickly changes it).

No you are supposed to write  $5\cos 25$  aren't you?

I: That is right. Before you work it out though can you estimate what the answer will be? Maybe you could draw a better diagram.

R: (She does not draw a diagram but looks at the paper). Probably about 4m but I'm not sure.



I: Well it is only an estimate, why not work it out and see how good an estimate it is?

R: (She picks up the calculator and works out the answer)  
Is it 4.5m?

I: Well you have rounded it to 1 decimal place again and the answer is right. It wasn't a bad estimate was it?

R: she laughs.

Although the interview went on to discuss other aspects of trigonometry it may be appropriate to compare this passage with the interview of Joanne from C1.

**Joanne, C1:**

I: This diagram is of a ladder or piece of wood leaning against a wall. Here is the hypotenuse, look, and it is 5m and the angle here is 25 degrees. I want you to try to find out this distance here.

R: Well you have to use that... (long pause)

I: What?

R: That theorem thing.

I: You mean Pythagoras'

R: Yes.

I: Well you need two sides before you can use Pythagoras, I have the angle and the hypotenuse.

R: Well in that case I can't remember.

I: suppose I wrote down sine 25 or cosine 25 (he writes)  
does that help you?

R: Oh yes.

I: so what do you remember about that?

R:  $\sin = \text{opposite over hypotenuse}$ ,  $\cosine = \text{adjacent over hypotenuse}$  and  $\text{tangent} = \text{opposite over adjacent}$

I: Good, you remember that, well done, now which side is which?

R: (she points and says) this is the opposite, this is the adjacent and this is the hypotenuse.

I: good, so lets write down an equation to help us find BC

R: she writes  $\sin 25 = BC/5$

I: well done, now how do we find BC from this?

R: I think I remember, times BC by 5 as well then we can cancel out the fives.

I: So we write down what?

R: she writes  $5 \times \sin 25 = BC$

I: Can you find the answer then (he hands over the calculator)

R: 0.435, but I don't think it can be right?

I: I think you realise that the answer is too small. (She laughs), let us look at what you put into the calculator. What should you put in first?

R:  $\sin 25$ ?

I: that is right now have another go

R: 2.11

I: Good, 2.11m. Now suppose I told you that before you came in I worked out AC to be 1.7m, what would you say to that?

R: Well I think that is about right.

I: How did you decide that?

R: Well I'm thinking about the ladder against the wall. Would it be longer?

I: Is anything wrong with my diagram?(long pause)How big is the angle in my drawing approximately?

R: What do you mean?

I: Well it says 25 degrees but what would you say I had drawn it as?

R: Oh. It is bigger isn't it? About 45 degrees.

I: Good, now can you draw me a better triangle for estimating the side? (she draws a triangle with a smaller angle) Now, how big do you think AC is now?

R: About 4?

I: Well done, now lets see how accurate your estimate was by calculating the answer.

R: Well its cos isn't it?

I: cos of what?

R: I'm not sure whether its 25 or 90

I: Why would it be 90?

R: When you change it you use the opposite angle, no you don't its 25

I: What will you write then?

R: (she writes  $\cos 25 = AC/5$ )

I: Now can we work out AC from this?



R: It is the same as the other one

I: well let us work it out.

R: (she takes her time and writes  $5 \times \cos 25 = AC$ , then uses the calculator). I make it 4.53.

I: Well done and your estimate was quite good wasn't it?

R: Yes it was all right.

The difference between the two girls in their ability to visualise the problem is quite marked. Joanne, from C1, is clearly caught up in the need for a procedure to the point where the question cues the Pythagoras frame before the trigonometry frame. She is able to work through the procedure with some prompting, but is unable to visualise the situation very well, and would have accepted the answer of 1.7m for AC. It appears that the confusion caused by an attempt to teach that the sine of an angle is the same as the cosine of the complement led to some doubt about whether the 90 degree angle or the 25 degree angle should be used moving from sine to cosine in the same triangle.

Caroline, from E1, had some initial doubts about her ability, answering "I don't know" to the first question but went on to display a good recall of the method and an immediate visualisation of the situation represented by the diagram.

The next student, James from E1, showed how nervous he was by again exclaiming that he couldn't remember, but after

some prompting was able to demonstrate a clear understanding.

James, E1:

R: (response to initial problem). It is to do with the opposite side.

I: How does that help us?

R: I can't remember really

I: I'll try and remind you. Let me draw you a diagram with the hypotenuse 1, what is BC?

R: Oh yes, it is sine 25

I: so what about the original diagram?

R: It is  $\sin 25$  times 5

I: O.K. can you work it out for me?

R: (He takes the calculator) It is 2.113

I: Good. now suppose I told you that before you came in I had worked out AC to 1.7m. What would you say to that?

R: (He turns the paper round and views the diagram) It can't be.

I: Oh, why not?

R: Because it must be larger than BC

I: How do you know that?

R: Well if it was smaller than AC the angle would have to be bigger than 45 degrees wouldn't it? The angle is smaller than 45 degrees so it must be bigger.

I: How big would you say AC was then, draw a new diagram if you like.

R: (He does not draw a new sketch, but looks at the diagram for a while and says) about 4.

I: Can you work it out using the calculator to see how good your estimate is?

R: It is  $5\cos 25$  isn't it? (without waiting for an answer he uses the calculator and says) it is 4.531. I was going to guess that it was more than four.

I: Well done James.

David, C1:

R: (response to original problem) Well you have to find the adjacent with the angles. I mean, well this is the adjacent, you have to find the opposite, then its opposite over hypotenuse.

I: So what would you write down?

R: (He writes  $\cos x$  then a fraction line under all of it then 5 underneath)

I: Is this all over 5 or just the x?

R: I don't know really, just the x I think.

I: And what is x?

R: Well it is the distance up.

I: How can this tell us the distance up the wall?

R: I can't remember



I: Let me try and help you. You can't have a cosine of a distance you see because the sine and cosine have to sine or cosine of an angle.

R: I see what you mean.

I: Now in this diagram what do you think we will be using the sine or cosine of?

R: I can't remember?

I: Well its related to the angle of 25 degrees. we need to use the sine or cosine of 25 degrees. Did you learn anything which would help you to remember when to use sine and then to use cosine? (A long pause). What about SOHCAHTOA?

R: Yes that's it

I: And what do we use it for?

R: I can't remember, just the word

I: Let me help you, did you learn that  $\sin = O/H$ .  $\cos = A/H$ ,  $\tan = O/A$ ?

R: Yes, I did, I remember that bit now.

I: So which of these is the opposite side?

(He points). That is right so we can write (the interviewer writes  $\sin 25 =$  )

R: h over 5

I: good, now can you use the calculator to find h?

R: No, not really, I can't remember what to do now

I: We multiply both sides by 5

R: Oh yes, I seem to remember that bit

I: O.K. Write it down eh.

(he writes  $5x\sin 25 = h$ ) Good, now you could use the calculator to find  $h$

R: (He shows the correct answer on the calculator)

I: Let us call that 2.11m shall we, well done. Now suppose I had previously worked out that AC was 1.7m, what would you say to that ?

R: Well you multiply them together and subtract that from that and look at the square root.

I: You are telling me something about Pythagoras' theorem I think, and that gives us a good method of working out AC when you have the other two sides. Can you just look at the diagram and tell me whether the answer looks reasonable?

R: You want me to work it out?

I: No not yet. Could you draw a sketch so that you might be able to tell whether the answer was reasonable?

R: What do you mean

I: Does this angle look like 25 degrees?

R: It looks a bit bigger

I: That is right, can you draw me one which looks better?

(he draws a more accurate sketch, though the angle is still very much larger than 25 degrees). Now what would you say the side was?

R: More than 3, probably about 4

I: Now If I write down  $\cos 25 = x/5$  and  $x$  is AC this time can you work it out?

R: Oh yes, (he uses the calculator to find a value of  $x$  of 113) I don't think that is right.

I: We need to find the cosine of 25 degrees first then multiply it by 5. (after some more help he produces 4.53m)  
Your estimate wasn't too bad then was it?

R: (He laughs)No. (pause)I need to revise this all again I think.

There is a contrast between James's initial inability to access the information in long term memory, when suffering from the anxiety of being interviewed and the confusion shown by David in his interview.

The third interview from E1 reveals problems with the use of the calculator and an initial agreement with the proffered answer for the length of BC.

**Hannah, E1:**

R: (response to the original problem). I think I would use the sine.

I: But you are not sure?

R: No, but I think so, I can't remember all that clearly.

I: Did you have a way of remembering which ratio to use?

R: Oh yes. SOHCAHTOA

I: So look at the diagram again-which side are we trying to find?

R: (She points) This one. the opposite.

I: That is right, now how will you find that length?

R: Sin 25 times 5



I: (Handing her the calculator) Can you do it?

(She produces the answer 4.09576, by multiplying 25 by 5, finding the sine and then, realising the answer was too small, she then multiplied it by 5. She actually produced  $5 \sin 125$ ). Why did you multiply the 25 degrees by 5?

R: Don't know

I: Let us have a look at this triangle. (He draws a triangle with hypotenuse 1 unit and angle 25 degrees). Now the hypotenuse is 1, what is BC?

R: It is the sine of 25 degrees

I: That is right, now what have I done to BC when I scaled the hypotenuse up to 5 in this triangle?

R: You have multiplied it by 5

I: Right, now can you find BC?

R: (She uses the calculator). It is 2.1m

I: That is right to one decimal place. Now, suppose I told you that before you came in I had worked out the length of AC to be 1.7m. What would you say to that?

R: It would be right.

I: Have a good look at the diagram, can you see anything misleading?

R: Oh, the angle is wrong?

I: You don't think it is 25 degrees?

R: No it is about 45 degrees.

I: Will you draw me a better sketch then?

(She does so). Now what do you think?

R: It is bigger than you said. (Pause)

I: If the angle was 45 degrees what would AC be?

R: It would be the same, 2.1m?

I: How big do you think AC actually is?

R: A bit more than 4

I: Can you calculate it and check?

R: It would be cosine wouldn't it?

I: That is right.

R: (She speaks as she does the calculation). Cosine 25 times 5 is 4.5m.

I: To one decimal place, that is right. Your estimate was pretty good wasn't it?

R: I suppose so.

Hannah clearly had some difficulty, but she responded well to each prompt. Her understanding of the sine of an angle was clear and her ability to estimate was quite good, though she did not demonstrate the immediate flexibility of being able to visualise the situation from the diagram. By contrast, Sarah from C1 is once more caught up in the difficulty of the procedure, though she shows an ability to visualise when her attention is drawn to the problem.

**Sarah, C1:**

R: (Response to original problem- Long pause)

I: Any idea at all?

R: You use the sine to find BC (She laughs)

I: That is right, how did you know that we use the sine?

R: Because it is opposite the angle, and because em (pause)

I: That is right, now can we write something down if you can't work it out in one go? The sine of?

R: 25 degrees

I: Is?(long Pause) The opposite, you were telling me

R: BC

I: O.K., over

R: 25?

I: 25?

R: No, 5

I: O.K. So we now have the sine 25 =  $BC/5$ , Do you remember seeing this sort of thing?

R: Yes I do now.

I: Can we find BC from this?

R: If we work out the sine of 25 and divide it by 5?

I: Divide by 5?

R: Yes.

I: Well try that

R: (She does) Oh no, you multiply by 5.

I: Good, you realised that it wouldn't be 5cm up the wall.(she laughs). Well let's do it correctly, what is BC

R: (She uses the calculator) 2.11?

I: Good, so BC is 2.11m, now suppose I told you that before you came in I had worked out the answer to AC and it came to 1.7m. What would you say to that? (Long pause). Does it seem like a reasonable answer?



R: I think it should be more.

I: What makes you say that?

R: I think if that is 5m then it will be more than that

I: What do you think is wrong with my diagram then? (Long pause, no answer). Well suppose I had not put 25 degrees in here for the angle, what would you say it was?

R: More than that, about 45 degrees.

I: Well can you draw me a diagram with the angle looking more like 25 degrees? (she draws a reasonable diagram). How big would you say BC was now?

R: I don't know

I: Just look at your diagram and think of the hypotenuse as 5, then estimate BC.

R: (Long pause) A bit smaller than 5?

I: Well let's calculate that length and see how good an estimate that is.

R: It is to do with cosine of 25.

I: Good.

R: It is 2.11 over 5, em-no it is 25 degrees then cosine, then times both sides by 5.

I: Well done. Can you do that?

R: It is 4.5m.

I: Good, your estimate of a little less than 5 was a good one wasn't it?

R: Yes. (she laughs)

The differences between the responses of Hannah and Sarah are not nearly so marked as with the other two pairs of subjects.

The next pair of responses from E2 and C2, illustrate the difficulty in assuming that assumed ability in mathematics is constant across all areas of mathematics. Lisa from E1 displays a sureness which was lacking in Hannah from E1.

Lisa, E2:

R: (Response to original problem). I'd put 25 in the calculator, then if I wanted that (she points to BC) I'd use Sine then I'd times it by 5.

I: Well done. Can you write it down for me? (She writes  $25\sin x \times 5$ ). Is that right now? That is what you do with the calculator but is it the correct expression?

R: No. You put 5 first don't you?

I: Right, so what would the equation be?

R: Oh yes (She writes  $5 \times \sin 25 = BC$ )

I: Good and what is the answer?

R: (She works it out) 2.11 I make it.

I: 2.11m. Well done. Now suppose I told you that before you came in I worked out the answer to AC and it came to 1.7m. What would you say to that?

R: No it would be bigger.

I: I wonder what makes you say that.?

R: Well if that is 25 degrees it would be bigger.

I: Why don't you draw me another diagram and see if you can make a guess at the size of AC?

R: (She draws a very good diagram). About 4 point something.

I: Can you do it for me and see what it is? (Without further prompting she produces 4.5m). Well done. Your estimate was pretty good.

The interviewer wished to discover how much of Lisa's quick response was based on a clear understanding of the concepts so the interview proceeded.

I: What told you that my original answer was too small?

R: The angle is smaller than 45 degrees.

I: What would happen if the angle was 45 degrees?

R: They are the same aren't they.

I: If I wanted a diagram where BC was exactly the sine of 25 degrees and AC was the cosine of 25 degrees like this. (He marks a sketch). What would have to be the important thing?

R: This has to be fixed. (She points at the hypotenuse)

I: Yes but fixed at what length?

R: It is just one isn't it. (Said with assurance rather than as a question)

I: Well done Lisa.

Again there is a clear contrast between this response and the next student.



Liz,C2

R: (Response to original problem) Well I'd put either 5 metres or 25 degrees then tan or something.

I: Let's slow down and think about what is given on the diagram and what we have to find out.

R: Oh no. You have got the angle haven't you?

I: Yes.

R: And you have got the hypotenuse, so- I don't know because usually we were given some of them and we had that as well. (She points to BC)

I: I see so if you were given two of the sides you could find the angle?

R: Yes I should think so-yes we were sort of-we were given sort of, how long the ladder was and em-I can't remember now.

I: Well what words do you associate with trigonometry? You mentioned tangent, are there any others?

R: Sine and cos.

I: Well what do they mean-what did you learn about them?

R: I know. Tan=opposite over adjacent, sine =opposite over hypotenuse, cosine = adjacent over hypotenuse.

I: Good, now let us look at the diagram. Now the sine of what is opposite over hypotenuse?

R: 25 degrees.

I: Good, let's write something down. (He writes  $\sin 25 = BC / \text{Hypotenuse}$ ). And what is the hypotenuse?

R: 5

I: So  $\sin 25 = BC/5$ , now how do we find BC from this?

R: Sin 25 divided by 5 or is it times by 5 oh I don't know.

(she laughs).

I: Well why don't you try it with the calculator?

R: (She performs the correct calculation) 2.11? No that can't be right. Is it right?

I: It is right.

R: Oh (she laughs)

I: Suppose I said that before you came in I had calculated AC and it came to 1.7m. What would you say to that?

R: It looks nearly right-that looks a little bit longer than that. (she claims that BC is actually slightly bigger than AC on my diagram) That would be cosine.

I: Well it certainly looks a little bit longer on my diagram but is my diagram a good one?

R: What do you mean?

I: Well I just drew any old angle here and marked it 25 degrees. What would you say the angle was?

R: Oh (surprised) it is not drawn to scale is it?

I: what would you say the angle was?

R: about 45 degrees?

I: Good , now could you draw me a better one? (She does but the angle is very small). What would you say AC was now?

R: Nearly 5m

I: So what size?

R: About 4.9m

I: Well done, now let us calculate AC and see how close you were.

R: It is cosine isn't it?

I: That is right.

R: (she performs the calculation well). It is 4.53m.

I. Well done, your estimate was good.

The next two pairs of interviews, (both pairs are boys), show less versatile thinking from the experimental group than had been displayed by the members of E1, though this is consistent with the resistance the researcher felt amongst some boys in this group to the approach being taken. However, the control subject displays confusion in all aspects of the interview.

**Richard, E2.**

R: (Response to original problem). I'll try and remember it. This is the hypotenuse and this is the adjacent (he points correctly). This is the opposite, so it is sine isn't it?

I: Does the diagram give you the information you need to work out BC?

R: It is O.K. It gives you the hypotenuse and the angle.

I: So how do you find the length BC?

R: I'm trying to remember actually. That's it you use Sine 25, it is sine isn't it?



I: Well you are right we use the sine of 25 degrees.

R: And for that you use cos. Then you times it by 5.

I: Can you write an equation then for BC?

R: Oh yes. Sine of 25. (He writes  $BC = \sin 25 \times 5$ )

I: And what does that give us?

R: (He uses the calculator) 2.11m.

I: Well done. Now suppose I told you that before you came in I had worked out AC to be 1.7m. What would you say to that?

R: Well that would be O.K.

I: Look at the diagram carefully. Is it a good one.

R: Yes you have put all the things on it.

I: Have a look at the angle. What would you estimate it to be?

R: Oh. about half of 90 degrees, about 45 degrees.

I: Can you draw me a better diagram. (He draws a good clear diagram). Put in the information again. What do you think of my answer now?

R: It is wrong.

I: How big do you think AC is then?

R: About 4.5m

I: Well let's work it out shall we?

R: This angle is 65 degrees.

I: You can use that or you can use the 25 degrees.

R: It is 25 cos, is that it, no we times it by 5. It is 4.53m.

I: Well your estimate was very good.

The researcher was a little perturbed by the way in which Richard operated the procedures but was slow in switching to a visualisation. He decided to probe to see whether the exposure to the experimental treatment had made any more of an impact than providing a means towards an efficient calculating procedure.

I: Suppose someone said "I understand what you have just done but I'm not really sure what is meant by sine 25 degrees" What would you say? Draw a diagram to try and show what you would say

R: (He draws a triangle, labels it ABC and marks the angle 25) It is the length BC?

I: Is it? Well it would be, but what is important if BC is exactly the sine of 25 degrees?

R: The hypotenuse.

I: What about it?

R: It has to be 1.

I: Well what if the sine of the angle here was the same as the cosine?

R: The angle would have to be 45 degrees.

I: Well done Richard.

Richard had clearly gained an understanding of sine and cosine of an angle and was able to visualise the sine being equal to the cosine when the angle is 45 degrees, despite

his initial hesitancy in moving into a visualisation of the original problem.

Nicholas from C2, displays more confusion and has great difficulty in visualising the problem .

**Nicholas, C2:**

R: (Response to original problem) Can you find the length of this by sine cos or tan?

I: Well, sine cos or tan of what?

R: 25 degrees.

I: Good. I want to find BC, the distance up the wall first.

R: Silly Old Harry Caught A Herring Trawling Off Aberdeen.

I: That is how you remember it is it? (They laugh)

R: That is the adjacent, so it would be cos.

I: Which one did you say was the adjacent?

R: No it is Opposite over Hypotenuse so its sin.

I: Let us do it then.

R: I can't remember any of this.

I: We need to find BC, so you have told me it is the opposite over the hypotenuse for the sine. So it is Sine of

R: BC.

I: No we can't have the sine of a side.

R: Sine of 25?

I: Is?

R: BC?

I: BC is what we want to find so it is BC over?



R: 5m is it?

I: That is right. (He writes it down) Now how do we find BC?

R: No, I can't do it.

I: Well we have got somewhere here. How can we find BC?

R: 25 degrees times 5?

I: No we have to multiply both sides by 5 to give  $\sin 25^\circ \times 5 = BC$  (he writes it down). Now can you work it out?

R: (with the calculator). So I have to do  $\sin 25^\circ$  then multiply it by 5?

I: That is it.

R: (He does so correctly) Is it 2.11?

I: 2.11m, that is right. Now suppose I had worked out AC before you came in and thought it was 1.7m. What would you say to that? (Long pause). Does it look right?

R: I think it is right yes.

I: What makes you come to that conclusion?

R: Well you square that and multiply it by that, which is about right.

I: Well that is interesting. You tried to calculate it. What about just looking at the diagram?

R: 1.7 is too long.

I: Too long? (Pause) Is my diagram an accurate drawing?

R: (Pause) No. It is not drawn to scale.

I: How big is this angle I've drawn

R: em-a bout 45 degrees?

I: Can you draw me a better diagram so that you can estimate AC?

R: (He draws a diagram, but the angle is close to 45 degrees again). It is about 3

I: I don't think your angle is much better than mine as a sketch of 25 degrees. Think of it as being smaller than 45 degrees.

R: (He draws a better sketch). About 4m?

I: Now let us work it out shall we?

R: It is cos isn't it?

I: So can we write it down? Cos 25 is?

R: AC over 5.

I: Which gives us  $AC = 5 \cos 25$ . Now work it out eh.

R: (He calculates correctly) 4.53m.

I: Your second estimate was much better.

Simon, from E2 has a similar set of responses to Richard from the same group. perhaps reflecting once more the lack of flexibility shown by the boys in this group. However, in comparison with Alan from C2, he appears to have a clearer understanding of trigonometry.

**Simon. E2:**

R: (Response to the original problem) To find that side I'd work out the sine of the angle and that side would be the cosine. (He points to the correct sides)

I: Would that be the answer then?

R: No but I'm not sure-I can't remember

I: How did you know that it was to do with the sine?

R: It is opposite and the cosine is the adjacent.

I: What is important about the hypotenuse if the opposite was the sine of the angle?

R: Oh yes you multiply by this.

I: So if it was the sine then the length of this (he points to the hypotenuse) would be?

R: One

I: That is right. Can we do this?

R: (he carries out the calculation) It is 2.11.

I: 2.11m, right. Now suppose that I had worked out the length of AC before you came in and it came to 1.7m. What would you say to that?

R: I don't know, I'm not sure.

I: Well look at my diagram, does it represent the situation? (pause) Look at the angle, how big is it approximately?

R: About 41 degrees.

I: And it is supposed to be?

R: 25 degrees.

I: So why not draw a better diagram. (He draws a very accurate diagram) Now can you say approximately how big AC is?

R: It is more than 3?

I: What would tell you whether AC was bigger than BC?



R: The actual size.

I: Would you be able to tell from the angle?

R: Well if it was smaller than 45 degrees I suppose.

I: That is right. Good, now here is the calculator, will you work out AC for me?

R: It is cosine of 25 x 5 isn't it? (without waiting for an answer) That is 4.53.

I: Good. So you were right, but it was quite a bit bigger than 3.

**Alan, C2:**

R: (Response to the original problem) Em-I'd do something with the angle I think.

I: What sort of thing would you do?

R: Would it be inverse sine?

I: Inverse sine? (Pause) Let us think why you wanted to use sine at all.

R: Well I'm trying to find BC so you would use sine.

I: Why?

R: It is the opposite over the hypotenuse I think.

I: Good, that is right. So sine = opposite over hypotenuse. But in this triangle we need to think of the sine of...what?

R: This side.

I: No you can't have the sine of a side can you?

R: No, the angle.

I: In this case 25 degrees. And the opposite in this case is?

R: You don't know that-it is BC.

I: Well done, and the hypotenuse is 5 isn't it? So we can write down  $\sin 25 = BC/5$ . (He writes it down) How do we find BC then?

R: You use cosine.

I: Well we don't need to think of the cosine for this equation because we can find the answer from what we have written down.

R: Is it  $\sin 25$  divided by 5?

I: No we need to multiply both sides by 5

R: So it is sine 25 times by 5?

I: That is exactly right. Let us do it.

R: (he uses the calculator) Is it 2.11?

I: that is right 2.11m, good. Now suppose I told you that before you came in I had worked out the answer to AC and it came to 1.7m. What would you say to that? (pause) Does it seem reasonable?

R: Yes it does. I think so

I: What makes you think so? Have a good look at the triangle. Does the angle look accurate?

R: Well if that is two eleven then this must be something like that-no if you times that and that together, No (long pause)

I: You are probably thinking about Pythagoras' theorem. Just have a look at the angle here. What have I drawn for angle A?

R: 25 degrees

I: Well it says 25 degrees, but what does it actually look like?

R: An acute angle.

I: If I hadn't put any figure there at all, what would you say it was?

R: You have to use inverse sine.

I: Can't you just guess what it looks like?

R: It looks about 25 degrees, it looks about right.

I: Well if I put a 90 degree line here, how big is angle A?

R: I don't know.

I: Well the problem is that in my diagram the angle marked as 25 degrees actually looks just a little bit less than 45 degrees, so the length of the sides in the diagram aren't a good indication of the sides in the problem. Try and draw a better one. (He spends some time) That is interesting because you have drawn a diagram that looks just like mine. You see the angle doesn't really look like 25 degrees. Let me draw a better diagram. Now what does AC look like now?

R: About 4 point something.

I: Now let us see if we can calculate it.

R: Is it sine?

I: We are finding a different side now.

R: Cosine?



I: That is right so AC over 5 is ?

R: The cosine of 25 degrees.

I: Good, so can we find AC?

R: It is cosine of 25 x5 which is 4.53.

I: Well done.

Alan showed the greatest inability to deal with estimates of angles. He was searching for numerical procedures and found it impossible to visualise the angle of 25 degrees.

The first pupil from E3, shows again, like Richard and Simon from E2, that he can calculate the sides and can visualise the problem well enough to draw a good diagram, but did not originally move into a visualisation of the problem from the information.

**Martin, E3:**

R: (Response to problem) You find  $5 \times 25 \sin$

I: You are right but how did you know that was true?

R: SOHCAHTOA

I: Oh you remembered that, good. So this side, BC is ?

R: The opposite.

I: How did you know that you had to multiply the sine by 5?

R: Because this is 5. (He points to the hypotenuse)

I: I wonder if you can remember when we started this and we had a triangle like this (he sketches a triangle) and this side was exactly the sine of 25 degrees, what must AB be?

R: It is always the longest. (pause)

I: Yes but what is its length?

R: Oh yes, 1.

I: One, that is right, and what would this side be? (he points to AC on the original drawing).

R: It would be 5 times cosine 25. (he corrected the order from his first statement, which was partly the order of entry in the calculator)

I: Well let's work out BC shall we?

R: (He does so and shows me the display)

I: That is close to 2.11m, well done, now suppose I said that I had worked out AC before you came in and it came to 1.7m. What would you say to that?

R: (Long pause, while he studies the diagram) I'm not sure.

I: Well how big is my angle approximately?

R: About 40 degrees.

I: So can you draw me a better one (he does so). What do you think of my answer now?

R: It is wrong, it must be bigger.

I: How big do you think it is?

R: About 4.2.

I: Well that is interesting, can you work it out?

R: (He works it out without hesitation) It is 4.5.

I: So your estimate was good. If the angle was 60 degrees, which would be bigger AC or BC?

R: BC

I: How did you know that?

R: Well this side gets squashed up when the angle gets bigger.

I: What if they were the same?

R: The angle would be 45 degrees.

I: Good, well done Martin.

A similar set of questions were tried with Clare from C3

Clare, C3:

R: (Response to initial problem). First of all you would have to label the sides, then you would have to see whether it is the sin or tan so it is hypotenuse and adjacent, no its not- it is hypotenuse though.

I: Let us concentrate on BC

R: Yeh it is hypotenuse and adjacent so its 25 sin which is adjacent over hypotenuse-no you go 5 times 25 (pause).

I: No Not really-can you write down an equation: Sine 25= ?

R: BC over 5?

I: Well done- that is right, so how do we find BC?

R: Divide by 5?

I: No

R: Then it is multiply by 5

I: So can you do that now with the calculator?

(She has two attempts, then shows the correct display).

That is right 2.11m. Now suppose I told you that before you



came in I had worked out the length of AC to be 1.7m, what would you say to that?

R: It is something to do with Pythagoras' theorem, no I can't really.

I: Well you are right we could work it out by using Pythagoras' theorem but I was thinking that just by looking at the diagram with 2.11 here and 25 degrees here would you think that 1.7 was reasonable?

R: Well it looks about right. I can't remember how to do it really.

I: Well look at the angle I have drawn, does it look accurate?

R: Yes.

I: You see I have marked it 25 degrees, but if I hadn't what would you say it was?

R: Oh. a bit more than that, more like 30.

I: Can you draw me a better diagram? (she draws a diagram, which is not too dissimilar from mine, the angle is about 35 degrees). How big is AC in your diagram?

R: More than 1.7, It is about 3.

I: Can you work it out with the calculator?

R: (Long pause, several different keys pressed but no communication). It works out the same as BC.

I: So do you think that AC and BC are the same?

R: Well it is the same diagram. Do you use sine?

I: Well didn't you use the sine before?

R: Oh no- I should have used tan

I: No you are using these two sides, the adjacent and the hypotenuse, so you use the cosine of the angle.

R: Is it 4.53.

I: That is it. We can see from your diagram that AC is bigger than BC. Can you tell which side will be the bigger by looking at the angle?

R: I don't think so?

I: Well if I drew an accurate diagram with 60 degrees here, would you be able to tell which side was bigger.

R: Not just by looking;

I: Well which angle here would mean that they were both the same?

R: Oh. 45 degrees

I: That is right so what would happen to the sides if the angle is made bigger than 45, like 60 degrees?

R: One of them would be bigger but I couldn't tell you which one.

I: O.K. Thanks Clare.

In the final pair from E3/C3, Fiona (E3) shows a clear understanding of how to calculate the answer and does so very quickly. She also displays a good visualisation of the problem followed by a good estimate. By contrast Kirsty (C3) had to be prompted all the way to the calculation which she attempts by dividing sin 25 by 5. She recognises the answer can't be right and suggests that she should have "timesed it". Both interviews will be quoted from the

response to the suggestion that the interviewer had previously calculated the answer to AC as 1.7m.

**Fiona, E3:**

R: What do you mean?

I: Would you agree with the answer?

R: Well probably not

I: What makes you say that?

R: Well it would be more than that.

I: Can you draw me a better diagram than this, with the angle drawn better? (She draws a very good diagram). How big would you say AC was?

R: About 4.5

I: Let us see how good an estimate that is. Can you work it out?

R: It is cos this time isn't it?

I: That is right.

R: It is 4.53.

I: So your estimate was very good wasn't it? (She laughs)

What would the angle have been if AC and BC were the same

R: (Without hesitation) 45 degrees.

I: Suppose the angle was made bigger than 45 degrees, what would happen?

R: (She points) This would get bigger, but this would get smaller.

I: That is right, well done Fiona.



**Kirsty, C3:**

R: I would say that was about right.

I: So if this was 2.11m up the wall and the angle was 25 degrees, then it would be 1.7m away from the wall?

R: Yes

I: What makes you arrive at that decision?

R: Well it has to be less than 5-the hypotenuse has to be bigger, so as long as it's less it's O.K.

I: Do you think the diagram is an accurate diagram, look at the angle.

R: Yes

I: Well look at my angle A which is marked as 25 degrees, do you think it looks like 25 degrees?

R: Yes.

I: Let me put a 90 degree line in here, now how big would you say the angle was

R: Oh about 45 degrees.

I: That is right now can you draw me a better diagram, with the angle more like 25 degrees and estimate the size of AC

R: (She draws a better diagram). It is bigger than 2.11, about 4?

I: Now let us try and work out how big this side is.

R: Is it tan?

I: No we are using these two sides, so it is adjacent over hypotenuse, which is cosine 25 degrees.

R: (She works it out as 4.53m)

I: Well done. How could you tell from looking at the angle here which of the two sides would be bigger? (No response) What would the angle be if they were the same? (No response). Let us sketch a triangle where they were the same. Now what is this angle?

R: 45 degrees?

I: Now if the angle got bigger than 45 degrees, which side would be bigger? (he demonstrates by rotating the pencil)

R: I don't know.

I: Look at the sketch, and imagine this pencil is the hypotenuse, now if this angle was getting bigger than 45 degrees, which side would be getting bigger? (he demonstrates by rotating the pencil).

R: BC.

I: That is right. Well done Kirsty.

The researcher was interested to know if the members of E4 could demonstrate an understanding of this topic, despite the fact that it was considered too difficult for them to study. Zoe, showed that she could perform the calculations with some prompting and was able to draw a better diagram, estimating AC to be about 4, though she needed a prompt to focus on the angle. She realised that BC would be bigger if the angle was changed to 60 degrees without having to sketch the figure. The interview ends with:

I: How would you know by looking at a diagram which side was going to work out to be the bigger?

R: Well if it is over 45 degrees BC would be bigger than AC.

Zoe showed a very clear ability to work with the visualisation of the situation, but even more impressive was Nicholas from E4.

The interview is quoted from the posing of the original problem.

**Nicholas, E4:**

R: Well this is the opposite, this is the adjacent and this is the hypotenuse, so it is the sine.

I: Sine of what?

R: Sine of 25 degrees.

I: Well is it exactly the sine of 25 degrees?

R: Oh yes. Times it by 5

I: Good, if this was exactly the sine of 25 degrees, what would this be? (pointing at the hypotenuse)

R: 1

I: That is right. (He points to the original diagram at AC). And what is this?

R: 5 times the cosine of 25 degrees

I: Well done, now can you work out AC for me?

R: (He presses some buttons) Eh? this can't be right. Oh yes you put the 25 in first (he presses some more buttons). It is 2.1



I: That is right, now suppose I told you that before you came in I had worked out AC to be 1.7m. What would you say to that?

R: No it can't be.

I: Why not?

R: Well if that is 25 degrees. to get a vertical line here. then this side has to stretch much further than 1.7.

I: Well what would you say it was?

R: Probably about 4

I: could you draw a better diagram and estimate AC

R: (He does so) It is about 4.3.

I: Work it out and see how good your estimate is.

R: (He does so without hesitation). It is 4.5

I: What if AC was 2.1 as well?

R: This would be 45 degrees.

I: If this angle was changed to 60 degrees which side would be bigger?

R: This one-BC.

I: Well done Nicholas.

Though some interview went beyond the transcripts quoted here, with members of E1 and C1 in particular questioned about maximum values of sine and cosine and introducing the value of the tangent for angles greater than 45 degrees, the basic pattern of responses did not differ from those quoted.

#### 10.4 Conclusion.

It is difficult for students to respond with ease to an adult interviewer, no matter how much the interviewer attempts to relax the interviewee. However, from the responses above certain tentative conclusions could be made about the ways in which the subjects approached the problem in hand. The problem itself was not unfamiliar to the students so it could not be claimed that the interview was assessing problem solving skills, but more the availability of supposed learned responses from long term memory. It would appear that the experimental subjects were now much more sure of the procedure for solving the original problem than the control subjects, which given the relative lack of confounding factors, would seem to support the proposition that the experimental treatment, based on the visual and numerical representation had allowed the students to develop more meaningful procedures. When certain students were asked to support their responses by referring to the conceptual understanding of sine and cosine of an angle they were able to do so.

By contrast many of the control group subjects, particularly those in C2 and C3, displayed much confusion about the actual procedure to be used. It could be assumed that for Nicholas, Alan, Clare and Kirsty, an attempt to learn a visually cued procedure in an instrumental way had led to immense problems when presented with a relatively

straightforward situation. No member of C4 could be assumed, from the results of post-test 1 and in the opinion of their teacher, to be able to use trigonometry at all so were not included in the interviews.

On the issue of versatile thinking the interviews reveal that one member of the control group, Sarah from E1, made an immediate limited response based a visualisation of the situation, and of the others three of the eight subjects made a good estimate when prompted to think about the angle in the diagram. The resistance to visualisation and inability to estimate angles demonstrated by Nicholas, Alan, Clare and Kirsty, from C2 and C3 was such that it was extremely difficult, even with prompting and sketching to encourage them to use a visual representation of the problem.

It could well have been deduced from the control groups that the ability to move between numerical and visual representation diminishes rapidly as general mathematical performance is reduced, with the better responses coming from members of C1. However, spontaneous visualisation, of the kind which did not occur with members of the control groups, was seen from Caroline and James from E1, Lisa from C2, Fiona, from E3 and Nicholas from E4. which would indicate that this facility is not restricted to students who had appeared to be of high mathematical ability in previous work. The other members of the experimental groups were able to estimate angles and sides



with some prompting, with no member of the experimental groups displaying the inability to visualise shown by the four control group members mentioned above. Richard and Simon, from E2 needed more prompting than the others but they were able to make estimates in a superior way than their counterparts from C2.

Though no claim is made that these interviews represent a controlled experiment, the results appear to support the findings of the post-tests and the suggestion that the experimental treatment, linking visual and numerical representation may lead to more versatile learning.

## 11. Conclusion and Implications For Teaching

### 11.1 Versatile thinking and mathematical procedures.

The results in both post-test 1 and post-test 2 led to a rejection of the null hypothesis that there will be no difference in performance between the students in the control groups and those in the experimental groups for all groups in the experiment. This would support the view that the linking of numerical and visual representation allowed the experimental students to gain a more global, visual understanding of trigonometry which allowed them to develop procedures successfully built on a conceptual understanding of the topic. The interviews gave further evidence of the limited visualisation skills of the control group subjects compared to the experimental group.

By dividing the tests into two separate categories, the more conceptual and the application of well practised procedures, it was possible to make further conclusions from the data. It is evident from both tests that the more able pupils can successfully develop procedures and recall them accurately from long-term memory, so that on these questions they scored almost as well as the experimental groups. despite the fact that there was a much poorer performance from the control group subjects in the more conceptual questions which had did not rely on reproducing these procedures. When the same approach is taken for children of

lesser ability, however, it can be seen that the ability to reproduce these procedures is significantly poorer for students in the control groups than for those in the experimental groups, with a complete breakdown occurring when the approach is taken with children below average ability (C4). The interviews again give support to these findings with students from C3 and C2, displaying confusion about the application of these procedures to a familiar problem.

It can be concluded from these findings that students of high ability can memorise procedures which are only instrumentally understood and recall them with some success, but that this facility is not evident with any but the most able children. With others, the building of conceptual base to the procedures, in Craik and Lockhart's (1972) model, the perception of 'meaningful' information, is a more successful way of encouraging the facility to recall and apply mathematical procedures.

There is evidence from this research, as there was in earlier research (Blackett N, 1987), that the ability to work with global, visualisations in mathematics is spread throughout the ability range, with some children in E4 showing a natural ability to visualise in short term working memory and, in the case of Nicholas (E4), to demonstrate the versatility of selecting appropriate visualisations or procedures from long-term memory, some eight weeks after the teaching sessions had ended. There are clear implications



here for attempts to raise the number of pupils studying mathematics to a higher level than G.C.S.E, indeed for any assumption that restricting syllabuses for 'average ability' by, for instance, removing trigonometry from the syllabus, is a straightforward matter. There should be some concern that students such as Nicholas are classified as requiring a more restricted syllabus when it is evident that the way in which they receive information has such an effect on their ability to perceive patterns and subsequently form conceptual frames.

Herbert H. Osborn (1983), in his study of 322 London fifth year pupils, found that it was possible to categorise the thinking involved in their mathematical activities into four distinct components. Osborn identified these components as: computational operations (C), pattern recognition (P), logical reasoning (L) and the symbolic manipulation of abstract quantities (S), and used the answers obtained from a bank of questions to give each pupil a mathematical profile in terms of the four components. Comparisons between the profiles of individuals in the sample and their results in G.C.E examinations led to the conclusion that the G.C.E. 'O' level examination is structured in favour of those with abilities in the L and S components. He states:

In view of the importance of the 'O' level examination in the social, notably, employment, life of school leavers, however, the question must be raised as to the adequacy of this structure as a basis of judging abilities on which the future of the individual depends. (p37)

Osborn's analysis of the demands of the G.C.E. examination on the most able children in our schools, concluding that success was associated with algebraic manipulation and logical serialistic thinking, illustrates the way in which these qualities were associated with good mathematicians. The evidence above, shows that symbolic manipulation and what appears to be logical reasoning may be shown by pupils in this ability range despite the fact that they may not be built on a relational understanding of the mathematics in question, which could be better developed by using the computer graphical approach incorporating the pattern recognition component (P). If the same skills were associated with the mathematical activity of children in lower ability classes then the approach is likely to very unsuccessful. As we have seen, the ability to recall these procedures accurately from long term memory when they are not built upon a conceptual base is particularly weak for children who are not classified as 'high ability' students.

The introduction of the G.C.S.E. courses with differentiated papers for assumed differences in ability will not provide a remedy for this situation if the same assumptions about what successful mathematical thinking involves are made, that is, success shown by higher ability children in reproducing procedures which may not necessarily be relationally understood encouraging a similar approach to be taken with children of lower ability.



It should also be a matter of concern that the number of children who study mathematics at a more advanced level after the age of sixteen may be relying on procedures which are not built into a strong conceptual frame to construct more advanced mathematical concepts. In this research, the poor performance in section A of the two post-tests shown by members of C1 when compared to members of E1, would appear to predict that the control group members will have difficulty in forming a conceptual understanding of the of trigonometric functions in pure mathematics as well as the extensive trigonometric work associated with applied mathematics. There is, too, the risk that having been able to reproduce procedures seemingly successfully in previous courses, students will continue to expect that all future mathematics can be learned in a similar manner. This would lead to yet further demands on teachers to reduce mathematics into procedural form and a temptation to mentally withdraw from attempts to place new mathematics in a wider conceptual frame.

Tall (1986) has shown that there often exists a great deal of confusion and misunderstanding in the minds of those who follow traditional calculus courses which may fall into the trap of asking students to identify skills in calculus with algebraic procedures. Similar problems occur in the teaching of calculus in the U.S.A., summed up by E.E. Moise's (1984) succinct statement:



For the overwhelming majority of students, the calculus is not a body of knowledge, but a repertoire of imitative behaviour patterns. (p37)

Osborn makes the pertinent point that teachers appear to have their own profiles in terms of the four components he associated with the mathematical thinking of fifth-year pupils. He suggests that these profiles had affected the way the teachers had acquired their knowledge and the way they "communicate that knowledge to pupils" (p37). He concludes that the methodology adopted by the teacher is liable to favour pupils with similar profiles, and by not allowing for differences, the teacher is liable to disadvantage pupils with strengths in other components. On studying the approach of student teachers he observes that this tendency is "...particularly noticeable in *computational* as opposed to the *pattern and spatial approaches*" (p37).

This observation has particular relevance to this research, for, the evidence is clear that the pattern and spatial element of the computer graphics, linked to the numerical representation, has distinct advantages over the more conventional computational procedural approach. For reasons given in chapter three, it appears that most cognition can not simply be identified as 'right hemisphere' as opposed to a 'left hemisphere' activity, and the point made by Gershen Rosen (1987) that both right hemisphere and left hemisphere approaches are possible with most topics in mathematics could be better expressed using Tall's

nomenclature of 'metaphorical right brain'. Nevertheless, the underlying assumption that, a more global visualisation of mathematical concepts can be integrated with the more serial step by step procedures to provide more versatile learning is the basis of this thesis, and Osborn's observations suggest that teachers tend to neglect the more spatial pattern forming approaches because their own mathematical profiles have been for a more computational, procedural foundation.

Whilst Osborn gives a clear account of teachers using methodology based on their own profiles he does not take into account the effect of the expectation of pupils on the methodology assumed by teachers. This point was made by the researcher when reviewing the results of earlier research on the teaching of linear and non-linear functions (Blackett N, 1989), and is evident in the comment by Mr. C, that though he knew that the students were not developing a conceptual understanding of trigonometry, they were demanding procedures which would enable them to solve problems and he found himself supplying these in order to maintain their belief that they were experiencing success. In the same way, the researcher found that some of the boys in E2 were anxious to receive short procedures to find answers and were reluctant to widen their conceptual frame. In these circumstances it should be recognised that it is difficult for teachers to withstand the pressure to produce instant



success, which in turn leads to the demand for more and more procedures to account for every eventuality.

When presented with questions which do not exactly cue the well learned procedure two prominent responses are noted: the first is that the student must have learned this procedure and forgotten it, and the second is that the question is unfair because the student had never been taught the appropriate procedure. Both of these responses were common in the post-reaction to post-test papers of members of the control groups and from the whole range of students involved in the pre-test.

If serious attempts are to be made to create more versatile learners it appears that teachers themselves have to appreciate the value of different approaches and that these approaches should be associated with mathematics from an early age. In this way it may be possible to change the expectations of pupils so that they do not view mathematics as the learning of procedures which do not need to be founded on a wider conceptual frame. If as, Osborn suggests, there are many secondary mathematics teachers who limit their methodology to their own profiles, it is likely that many primary teachers, who are not necessarily mathematics specialists, are similarly limited in their approach to teaching mathematics. If inroads are to be made into widening the approaches taken in order to encourage more versatile learning, then it is difficult to see how this can



be achieved without extensive in-service provision being made available.

#### 11.2 The effect of the experimental treatment on the performance of girls.

The results of both post-tests show that the improvement in the performance of the girls in the experimental groups in comparison with the girls in the control groups was greater than that of the boys in the experimental groups in comparison with the boys in the control groups. Indeed there were statistically significant differences on post test 2 between the girls in the experimental groups and those in the corresponding control groups for all three pairs of groups tested, and the results from post-test 1 show a similar significant difference for the fourth pair (E4,C4) which would almost certainly have been reproduced in the second test had it been thought worthwhile for C4 to sit this test. The boys in the experimental groups had superior results to those in the control groups, but the only statistical significance occurred in post-test 1, for the groups E3/C3 and E4/C4. (E3/C3 were close to significance in post-test 2 and it could be assumed that the boys in E4 would have outperformed the boys in C4). The general pattern was that the girls performed better than the boys in the experimental groups (with the difference being significant for E1), and the boys performed better than the

girls in the control group (with the difference being significant for C2.)

The matched pairs test also confirmed a significant difference in the performance of the girls in the matched pairs, but the poor performance of one of the experimental group boys led to a lack of significance in the boys' results.

In the light of the evidence on the relative performance of girls and boys in examinations at sixteen plus and the numbers of girls who choose not to study mathematics beyond this stage, (Chapter five), these results should have important implications for the teaching of mathematics. The analysis of examination questions undertaken by Wood (1976) and Bradberry (1989) and the clusters of tests carried out by the Assessment of Performance Unit (1980, 1981) may lead to the conclusion that girls display weaknesses on topics associated with visuo-spatial ability and ratio, particularly similar triangles, yet these concepts are crucial to the approach taken with the experimental groups. It would appear that the hypothesis set out in chapter five, that it is possible to increase the performance of girls in these areas by linking spatial and numerical coding to provide a conceptual basis for procedures, utilising the possibly superior skills of females in this cognitive flexibility and the ability to attend to varied perceptual stimuli, has been borne out by the results.



The analysis of examination and test results, both here and in the U.S.A., also suggest that it is at the highest levels of ability where the higher success rate of boys is most evident. However, in this research the position is completely reversed for the experimental groups, with the girls in E1 showing a statistically significantly better performance than the boys in the same groups and a very high degree of mastery of the topic (seven of the seventeen girls achieved full marks in post-test 2).

It would appear from the results, and certainly it was the subjective opinion of the researcher when teaching this group, that building the procedures on a conceptual basis had a remarkable confidence boosting effect on the girls in this group, which allowed them to believe that they could in fact 'master' the topic.

There has been some suggestion (Shuard H, 1981) that, though the A.P.U. primary surveys report very little difference in performance between boys and girls, as did the Schools Council Survey (Ward M, 1979), the fact that girls performed better than boys in certain computational skills leads to the conclusion that they are developing low-level procedural skills in the primary school, whereas boys are more likely to develop the higher level reasoning skills. In her exploration of the the issues surrounding girls and mathematics, Valerie Walkerdine (1989) lays great emphasis on the ways in which developments in teaching have changed the emphasis on what is considered to be good mathematics.



with the skills suggested as being superior in primary school girls being downgraded:

What is particularly contentious, however, is the downgrading of certain aspects of Mathematics as not 'real', or as only 'rule following', and upgrading others. It so happens, of course, that since girls do well at 'low-level' aspects, there is a failure to point out that they also perform well on others considered 'real'. It is a moot point whether rule-following should be considered low-level at all or whether 'real understanding' plays the part in school Mathematics which some Mathematics educators would have us believe. (p17)

Walkerdine has mixed several issues in this statement but it appears that she does not differentiate between procedures based on conceptual framework and those which are simply 'rule following' in an instrumental way. The latter is exemplified in this research by those students who could perform well on the routine questions in post-test 1, but were unable to understand the conceptual basis of trigonometry. In all but the most able, this resulted in a poorer performance in even the most routine 'rule following' compared with the experimental groups by the time the second post-test was given. Given the the superior performance of the girls in the experimental groups in the conceptual and the routine elements of the tests in this research, it would be unfortunate if Walkerdine's views were to lead to the belief that 'rule-following' ought to be more highly regarded, or that this skill should be associated with the performance of girls in mathematics.

The evidence of this research would suggest that girls' performance in mathematics could be improved, with the likely effect of more girls studying mathematics at an advanced level, if the information processing strategy, incorporating visual pattern detection, linked to numerical or algebraic procedures to develop versatile thinking, were used to build a wider conceptual framework for mathematical topics. Walkerdine's point that procedural thinking ought to be more highly regarded is one which this research would refute, but any assumption that girls are more prone to this kind of approach by any kind of neural organisation theory can not be supported by the results of this research, which suggest the very reverse of this assumption to be the case.

### 11.3. The use of computers in mathematics

The success in the teaching of the experimental groups was thus dependent on three interrelating factors:

1. The availability of appropriate hardware in a suitable location. In this case the appropriation of three B.B.C. Masters with disc drives to be available for every mathematics lesson during the two week period.

2. The availability of appropriate software; in this case designed to encourage a link between numerical and visual representation, and



3. The adoption of a teaching style which allowed the use of the computers to be effective in a suitable teaching environment.

These three equally valuable factors need to be addressed before computers make an impact on mathematics teaching. As stated in chapter one, the general picture of computer usage, which was reported by the Mathematical Association sub-committee (1987), is not encouraging. A contributing factor to this position would appear to be that the purchasing and distribution of computing facilities has not been undertaken with specific use of computers in the mathematics classroom in mind. It is difficult to predict at this stage whether the emphasis on data bases and spread sheets in the national curriculum will support the use of computers as a tool to develop concept acquisition in the way that they were used in this research, or whether schools will adopt the allocation of computer network time to learn about computers in a more general way, the computer becoming a topic to be learned about but not used to aid concept development in mathematics.

There are practical problems with the use of computers in the way they were used in this research, such as how they can be situated for maximum advantage and how they can be stored safely, which can create complications for mathematics teachers. It would be true to say that many mathematics classrooms were not designed to facilitate the use of computers. In this research the room used was



unusually large for the school and even so the problems involved with extension cables and sockets would have discouraged many teachers. Difficulties of this sort contribute to the "resistance" noted by the Mathematical Association Sub-Committee.

The software used in this research was designed by the researcher, its strengths had been assessed and the way it could be used to effect had been planned. The variation in available software, both in quality and design is a problem which affects the use of computers in the classroom. The report of the D.E.S. funded seminar for educationalists interested in computers in mathematical education. "Will Mathematics Count" (Ball D., et al, 1987) states:

The computer will only become a valuable component of a learning environment if the software offered to the learners is chosen carefully. Much of the software available is of little use, and such material has led many teachers to doubt the educational usefulness of computers. (p24)

Perhaps the only way this can be remedied is by mathematics educators, who themselves have an appreciation of versatile learning, producing software or presenting designs to programmers. A small number of programs developed in this way could have great effect over a range of topics and age ranges, particularly if the software was designed to provide a general purpose catalogue of activities which could be used primarily to develop concepts by allowing the student to generate their own inputs.

The results in this thesis add to a body of information including the results of the researcher's M.Sc. thesis (Blackett N, 1987), the research by D.O.Tall (1986) with sixth formers on the understanding of calculus, and that by M. O. J. Thomas (1988) on the introduction of variables in algebra with second year pupils, all of which show the value of computer graphics in developing conceptual understanding. The results quoted in this research provide evidence of the encouragement of flexible, versatile thinking across the ability range and in particular with girls, by the use of dynamic computer geometry linked to numerical representation. Despite the problems noted above it is clearly possible to encourage such learning in mathematics classrooms, but it is likely to require not only the in-service training advocated earlier but carefully planned development of resources and logistics within schools to allow teachers to feel empowered by computers rather than being beset by practical problems.

#### **11.4 Further research.**

This research was on a small scale and was limited to one school. However, the results obtained have produced some substantial implications for teaching.

The issue of versatile thinking, moving between visualisation and serialistic procedures, being improved by perceiving and relating visual and numerical patterns with a

computer is an exciting aspect of mathematics, which could have applications with different age ranges and across a range of topics. Further research, on a larger scale could bring new insights into the way children develop in teaching environments incorporating this approach. It would be particularly interesting to formulate a longitudinal study to discover the long term implications of exposure to this kind of teaching

The results of this research with respect to the performance of girls provides some encouraging evidence of a means to increase the number of girls achieving high levels of performance in mathematics. The fact that the computer based teaching benefited girls more than boys contradicts a popular held view that computers have a more positive effect on boys than girls. However, there is a need for more research to see whether the findings are replicated. It would be heartening to discover that the results found in this research were not a singular case, but represent an important foundation for improving the relative performance of girls in mathematics, and increasing the numbers who feel confident about studying the subject to an advanced level.



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